Superposition Operator on Some 2-Normed Sequence Spaces


c_{2,0} := \{ x = (x_i) \in w_X : \lim_{i \to \infty} \| x_i, z \| = 0 \text{ for all } z \in X \}.

It is easy to check that these spaces are Banach space with the norms

\| x \|_p = \sum_{i=1}^{\infty} \| x_i, z \|^{\frac{1}{p}} \| x \|_\infty = \sup_{i} \| x_i, z \|, \text{ for every } z \in X, \text{ respectively.}

The 2-normed spaces and more general n-normed spaces were studied by (Kim, Cho and White 1992), (Lewandowska 1999), (Lewandowska 2001) and many others. Recently, (Duyar et. al. 2016, Duyar et. al. 2017) studied on these spaces.

Let \( \lambda \) and \( \mu \) be two sequence spaces. Recall that a superposition operator \( P: \lambda \to \mu \) is defined by \( P(x) = (g(k, x_i)) \), where \( g: \mathbb{N} \times \mathbb{R} \to \mathbb{R} \) is a function such that \( g(k, 0) = 0 \). The superposition operators were studied F. Dedagich (Dedagich 1987), S. Petrantuarat and Y. Kemprasit (Petrantuarat and Kemprasit 1997), E. Kolk (Kolk 2004), B. Sağır and N. Güngör (Sağır and Güngör 2015), O. Oğur (2017) and others.

In this paper, we study on superposition operator \( P_z \) acts from the spaces \( c_{2,0}, \ell^{2,\infty} \) and \( \ell^{p,1}, 1 \leq p < \infty \), to the space \( \ell^{1,1} \) such that \( P_z(x) = (g(k, x_i)) \), where \( g \) acts from \( \mathbb{N} \times X \) to \( X \).

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2. Results

Throughout the results, we assume that the function $g(k,\cdot)$ is bounded on bounded subsets of $X$ for every $k \in \mathbb{N}$.

**Theorem 1.** $P_c: \ell^{2,\infty} \to \ell^{2,1}$ if and only if there exists $c = (c_i) \in \ell^{2,1}$ such that for every $\alpha > 0$ and for each positive integer $k$

$$\|g(k,t),y\| \leq \|c_i,y\| \quad \text{whenever} \quad \|t,y\| \leq \alpha$$

for all $y \in X$.

**Proof.** Let $P_c: \ell^{2,\infty} \to \ell^{2,1}$. For arbitrary $\alpha > 0$ and $k \in \mathbb{N}$ let define following sets;

$$A(\alpha) = \{t \in X; \|t,y\| \leq \alpha \quad \text{for every} \quad y \in X\} \quad \text{and} \quad B(k,\alpha) = \sup \{\|g(k,t),y\| \mid t \in A(\alpha)\}.$$

We need to show that $(B(k,\alpha)) \in \ell^{2,1}$ for every $\alpha > 0$. Suppose the contrary, i.e. there exists $\alpha > 0$ such that $(B(k,\alpha)) \notin \ell^{2,1}$. It means that

$$\sum_{k=1}^{n} B(k,\alpha_i) = \infty.$$ 

Thus, there exist natural numbers $n_i$, where $n_0 = 0 < n_1 < n_2 < \ldots < n_i < \ldots$, such that

$$\sum_{k=n_{i-1}+1}^{n_i} B(k,\alpha) > 1 \quad \text{and} \quad \varepsilon_i > 1.$$ 

(2.1)

Take $i \in \mathbb{N}$. Then, we have $0 \leq B(k,\alpha_i) < \infty$ for every natural numbers $k \in [n_{i-1}+1,n_i]$. By definition of $B(k,\alpha)$, there exists $x_i \in A(\alpha_i)$ such that

$$\|g(k,x_i),y\| > B(k,\alpha_i) - \varepsilon_i.$$ 

(2.2)

Thus, using (2.1) and (2.2), we have

$$\sum_{k=n_{i-1}+1}^{n_i} \|g(k,x_i),y\| > \sum_{k=n_{i-1}+1}^{n_i} B(k,\alpha) - \sum_{k=n_{i-1}+1}^{n_i} \varepsilon_i > 1$$

for every $y \in X$, which shows that $g(k,x_i) \notin \ell^{2,1}$. Also, since $x_i \in A(\alpha_i)$ for all $k \in \mathbb{N}$, we write $x_i,y \leq \alpha_i$ for every $y \in X$, which implies that $(x_i) \in \ell^{2,\infty}$. This contradicts with the assumption $P_c: \ell^{2,\infty} \to \ell^{2,1}$.

Conversely, assume that there exists $c = (c_i) \in \ell^{2,1}$ such that for every $\alpha > 0$ and for each positive integer $k$

$$\|g(k,t),y\| \leq \|c_i,y\| \quad \text{whenever} \quad \|t,y\| \leq \alpha$$

for all $y \in X$. Take $(x_i) \in \ell^{2,\infty}$. Then, there exists a real number $M > 0$ such that $\|x_i,y\| \leq M$ for every positive integer $k$ and all $y \in X$. By assumption, we have $c = (c_i) \in \ell^{2,1}$ such that $\|g(k,x_i),y\| \leq \|c_i,y\|$ for every positive integer $k$. Thus, we have

$$\sum_{k=1}^{\infty} \|g(k,x_i),y\| \leq \sum_{k=1}^{\infty} \|c_i,y\| < \infty$$

which shows that $P_c: \ell^{2,\infty} \to \ell^{2,1}$.

**Example 2.** Let $X = \mathbb{R}^2$ and let define a 2-norm on $X = \mathbb{R}^2$ such that $\|x,y\| = \|x_1,y_1-x_2,y_2\|$, where $x = (x_1,x_2),y = (y_1,y_2) \in \mathbb{R}^2$. Also, let define a function $g: \mathbb{N} \times \mathbb{R}^2 \to \mathbb{R}^2$ such that $g(k,t) = (1,0),\ldots,(t_i,0)$, where $t = (t_1,t_2) \in \mathbb{R}^2$. Given $\alpha > 0$ and $t \in \mathbb{R}^2$ such that $\|t_1,z\| \leq \alpha$ for all $z \in \mathbb{R}^2$. Then, since it is true for $z = (0,1) \in \mathbb{R}^2$, we write $\|t,1-t_2,0\| = |t_1| \leq \alpha$.

Thus, we have

$$\|g(k,t),z\| = \left|t_1 \frac{1}{4^{k+1}} - 0.3 \right| \leq \frac{\alpha}{4^{k+1}}$$

for every $z \in \mathbb{R}^2$. Choose $c_1 = (\frac{\alpha}{4^{k+1}},0) \in \ell^{2,1}$ for every $k$ and so $(c_1) \in \ell^{2,1}$. By theorem 1, we get $P_c: \ell^{2,\infty} \to \ell^{2,1}$.

**Theorem 3.** $P_c: c_{2,0} \to \ell^{2,1}$ if and only if there exist a real number $\alpha > 0$ and $(c_i) \in \ell^{2,1}$ such that

$$\|g(k,t),y\| \leq \|c_i,y\| \quad \text{whenever} \quad \|t,y\| \leq \alpha$$

for every positive number $k$ and every $y \in X$.

**Proof.** Let $P_c: c_{2,0} \to \ell^{2,1}$. Define the following sets;

$$A(\alpha) = \{t \in X; \|t,y\| \leq \alpha \quad \text{for every} \quad y \in X\} \quad \text{and} \quad B(k,\alpha) = \sup \{\|g(k,t),y\| \mid t \in A(\alpha)\}$$

for arbitrary $\alpha > 0$ and every $k \in \mathbb{N}$. By definition of these sets, we have $\|g(k,t),y\| \leq B(k,\alpha)$ whenever $t \in A(\alpha)$. Let suppose that $(B(k,\alpha)) \notin \ell^{2,1}$ for every $\alpha > 0$. Then, we write $\sum_{i=1}^{\infty} B(k,2^i) = \infty$ for every $i \in \mathbb{N}$. Thus, there exists a subsequence $n_0 < n_1 < n_2 < \ldots < n_i < \ldots$ such that

$$\sum_{k=n_{i-1}+1}^{n_i} B(k,2) > 1 \quad \text{and} \quad \text{positive real number} \quad \varepsilon_i > 1.$$ 

(2.3)

Take $i \in \mathbb{N}$. Then, for every $k \in \mathbb{N}$ with $n_{i-1}+1 \leq k \leq n_i$ we have $0 \leq B(k,2^{i-1}) < \infty$. By definition of $B(k,2^{i-1})$, there exists $x_i \in A(2^{i-1})$ such that

$$\|g(k,x_i),y\| > B(k,2^{i-1}) - \varepsilon_i.$$ 

(2.4)

By (2.3) and (2.4), we get

$$\sum_{k=n_{i-1}+1}^{n_i} \|g(k,x_i),y\| > \sum_{k=n_{i-1}+1}^{n_i} B(k,2^{i-1}) - \sum_{k=n_{i-1}+1}^{n_i} \varepsilon_i > 1,$$

which shows that $(g(k,x_i)) \notin \ell^{2,1}$. Also, since $(x_i) \in A(2^{i-1})$ for every $k \in [n_{i-1},n_i]$ we have $\|x_i,y\| \leq 2^{i-1}$ and so $(x_i) \in c_{2,0}$. This contradicts the assumption that $P_c: c_{2,0} \to \ell^{2,1}$.

Thus, there exists $\alpha > 0$ such that $(B(k,\alpha)) \in \ell^{2,1}$. By choosing $c_1 = B(k,\alpha)$, the necessity is completed.
Conversely, assume that there exist a real number \( a > 0 \) and \((c_i) \in \ell^{21}\) such that \( \|g(k,t),y\| \leq c_i\|y\| \) whenever \( \|t,y\| \leq a \) for every positive number \( k \) and every \( y \in X \). Let \( x = (x_i) \in \ell^{20} \). Then, for this \( a > 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( \|x_i, y\| \leq a \) for every \( k \geq n_0 \). By assumption, there exists \( c = (c_i) \in \ell^{21}\) such that \( \|g(k,x_i),y\| \leq c_i\|y\| \) for every \( k \geq n_0 \). Thus, we get
\[
\sum_{k=n_0}^{\infty} \|g(k,x_i),y\| \leq \sum_{k=n_0}^{\infty} c_i\|y\| \leq \sum_{k=n_0}^{\infty} c_i\|y\|
\]
which shows that \( (g(k,\cdot),y) \in \ell^{21}. \)

**Example 4.** Let \( X = \mathbb{R}^2 \) and let take a 2-norm on \( X = \mathbb{R}^2 \) such that \( \|x,y\| = \|x_1y_2 - x_2y_1\| \), where \( x = (x_1,x_2), y = (y_1,y_2) \in \mathbb{R}^2 \). Also, let define a function \( g: \mathbb{N} \times \mathbb{R}^2 \to \mathbb{R}^2 \) such that \( g(k,t) = \frac{(t_1, (t_2 - 1)^2)}{2^2}, 0 \), where \( t = (t_1,t_2) \in \mathbb{R}^2 \). Given arbitrary \( t = (t_1,t_2) \in \mathbb{R}^2 \) such that \( \|t_1,t_2\| \leq 1 \). Here we chose \( \alpha = 1 \). By definition of 2-norm, last inequality is true for \((0,1) \in \mathbb{R}^2 \). Thus, we have
\[
\|g(t_1,t_2),(0,1)\| = \|t_1\| \leq 1.
\]
Therefore, we get
\[
\|g(k,t),y\| = \left| \frac{t_1(t_1-1)}{2^2} y_1 - 0 y_1 \right| \leq \left| \frac{t_1+1}{2^2} y_2 - 0 y_1 \right| \\
\leq \left| \frac{2}{2^2} y_2 - 0 y_1 \right|
\]
where \((y_1, y_2) \in \mathbb{R}^2 \). If we choose \( c_k = \frac{1}{2^2} \), then \((c_k) \in \ell^{21}\). By theorem 3, we get \( P: \ell^{20} \to \ell^{21} \).

**Theorem 5.** \( P: \ell^{2p} \to \ell^{21} \) ifandonlyifthereexist \( a > 0, \beta > 0 \) and \((c_i) \in \ell^{21}\) such that \( \|g(k,t),y\| \leq c_i\|y\| + \beta \|t,y\| \) whenever \( \|t,y\| \leq \beta \) for every \( y \in X \). Take \( x = (x_i) \in \ell^{2p} \).

Since \( \sum_{k=n}^{\infty} x_i y \| < \infty \) for every \( y \in X \), we have \( \lim_{k \to \infty} \|x_i y\| = 0 \). Specially, for this positive number \( \beta > 0 \), there is \( i \in \mathbb{N} \) such that \( \|x_i, y\| \leq \beta \) for \( k \geq i \).

Then, by assumption, there exists \((c_i) \in \ell^{21}\) such that \( \|g(k,x_i),y\| \leq c_i\|y\| + \alpha \|x_i, y\| \) for \( k \geq i \). Thus, we get
\[
\sum_{k=n}^{\infty} \|g(k,x_i),y\| \leq \sum_{k=n}^{\infty} c_i\|y\| + \alpha \sum_{k=n}^{\infty} \|x_i, y\| \\
\leq \sum_{k=n}^{\infty} c_i\|y\| + \alpha \sum_{k=n}^{\infty} \|x_i, y\| \\
< \infty
\]
which shows that \( P: \ell^{2p} \to \ell^{21} \). Conversely, let \( P: \ell^{2p} \to \ell^{21} \) and given arbitrary positive numbers \( a > 0, \beta > 0 \). Let define the following; for every \( k \in \mathbb{N} \) and \( y \in X \)
\[
A(k,a,\beta) = \{ t \in X : \|t,y\| \leq a \}, \\
B(k,a,\beta) = \{ t \in X : \|t,y\| \leq \beta \}
\]
and
\[
B(k,a,\beta) = \sup \{ \|g(k,t),y\| : t \in A(k,a,\beta) \}.
\]
Then, we have \( \|t,y\| \leq \beta \) and \( \|g(k,t),y\| \leq B(k,a,\beta) \) whenever \( t \in A(k,a,\beta) \). Also, if \( \|t,y\| \leq \beta \) and \( t \in A(k,a,\beta) \), we have \( \|g(k,t),y\| \leq a \|t,y\| \). Thus, we can write that \( \|g(k,t),y\| \leq B(k,a,\beta) + a \|t,y\| \) whenever \( t \in A(k,a,\beta) \). We want to show that there exist \( \alpha > 0, \beta > 0 \) such that \( (B(k,a,\beta)) \in \ell^{21} \). By the assumption, we have \( \sum_{k=n}^{\infty} B(k,2^i,2^{-i}) = \infty \) for all \( i \in \mathbb{N} \). Then, there exist a subsequence \( n_0 = 0 < n_1 < n_2 < \ldots < n_i < \ldots \) such that \( \sum_{k=n_0}^{\infty} B(k,2^i,2^{-i}) > 1 \) for all \( i \in \mathbb{N} \) and positive number \( \varepsilon > 0 \) such that
\[
\sum_{i=n_0}^{\infty} B(k,2^i,2^{-i}) - (n_i - n_{i-1}) \varepsilon, i > 1.
\]
Thus, we have that \( 0 \leq B(k,2^i,2^{-i}) < \infty \) for all \( k \in \{n_{i-1} + 1, n_i \} \). By definition of \( B(k,2^i,2^{-i}) \), there exists \( x_i \in A(k,2^i,2^{-i}) \) such that
\[
\|g(k,x_i),y\| > B(k,2^i,2^{-i}) - \varepsilon,
\]
for all \( y \in X \). Using inequalities (2.5) and (2.6), we have
\[
\sum_{k=n_0}^{\infty} \|g(k,x_i),y\| > \sum_{k=n_0}^{\infty} B(k,2^i,2^{-i}) - \sum_{k=n_0}^{\infty} \varepsilon, i > 1
\]
for all \( i \in \mathbb{N} \), which shows that \( (g(k,x_i)) \in \ell^{21} \). Also, since \( x_i \in A(k,2^i,2^{-i}) \) for all \( k \in \{n_{i-1} + 1, n_i \} \), we have
\[
x_i \|y\| \leq 2^i \|x_i\| \|y\| \leq 2^i \|g(k,x_i),y\|
\]
for all \( y \in X \). Using the fact that \( \sum_{k=n_0}^{\infty} B(k,2^i,2^{-i}) \leq 1 \), we get
\[
\sum_{k=n_0}^{\infty} \|x_i y\| = \sum_{k=n_0}^{\infty} \|x_i, y\| + \|x_i, y\| \\
\leq \sum_{k=n_0}^{\infty} \|x_i, y\| \|y\| + \|x_i, y\| \\
\leq \sum_{k=n_0}^{\infty} \|g(k,x_i),y\| \|2^{-i}\| + \|2^{-i}\| \\
\leq \frac{1}{2^{-i}} \sum_{k=n_0}^{\infty} B(k,2^i,2^{-i}) + 2^{-i}
\]
which shows that \( (x_i) \in \ell^{2p} \). This contradicts the assumption that \( P: \ell^{2p} \to \ell^{21} \). By choosing \( c_k = B(k,a,\beta) \), we complete the proof.
3. References


Kolk, E. 2004. Superposition operators on sequence spaces defined by \( \varphi \)-functions, *Demonstratio Math.*, 37: 159-175.


Sağr, B., Güngör, N. 2015. Continuity of superposition operators on the double sequence spaces \( L_\mu, F ilomat, 29(9): 2107-2118. \)