On Certain Properties of Quadrapell Quaternions

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In [6], given a sequence $A = \{a_n\}_{n=0}^\infty$, the binomial transform $B$ of a sequence $A$ is denoted by $B(A) = \{b_n\}$ and defined by

$$b_n = \sum_{i=0}^{n} \binom{n}{i} a_i.$$

Recently, Kızılateş et al. [4] have been investigated some properties of binomial sums of quadrapell sequences and quadrapell matrix sequences. At this point, we refer the reader to [1, 2, 5, 6] for more details and interesting properties of the binomial transformation and its generalizations.

A quaternion $p$ with real components $a_0, a_1, a_2, a_3$ and basis $1, i, j, k$ is a hypercomplex number of the form

$$p = a_0 + a_1 i + a_2 j + a_3 k, \quad (a_0, 1 = a_0)$$

where

$$i^2 = j^2 = k^2 = -1,$$

$$ij = -ji, jk = i = -kj, ki = j = -ik.$$

Motivated by the above definition, A. F. Horadam [3] defined the $n$th Fibonacci and $n$th Lucas quaternions as follows:

$$Q_n = F_n + F_{n+1} + F_{n+2}j + F_{n+3}k.$$
and
\[ K_n = L_n + L_{n+1} + L_{n+2} + L_{n+3}, \]
where \( F_n \) and \( L_n \) are the \( n \)th Fibonacci and \( n \)th Lucas numbers respectively.

Inspired by the above studies, in this paper, we introduce quadrapell quaternions. We give some fundamental properties of these quaternions. In addition, we apply the binomial transform to these quaternions and derive some algebraic properties.

### 2. Fundamental Properties of Quadrapell Quaternions

In this section, we give a definition, the generating function, a Binet-like formula, and some summation formulas for the quadrapell quaternion sequence.

**Definition 2.1.** For \( n \geq 0 \), \( n \)th quadrapell quaternion is defined by
\[
w_n = D_n + D_{n+1} + D_{n+2} + D_{n+3}
\]
where \( D_n \) is the \( n \)th quadrapell number.

**Proposition 2.1.** For \( n \geq 0 \), quadrapell quaternion sequence satisfies the following recurrence relation:
\[
w_{n+4} = w_{n+2} + 2w_{n+1} + w_n
\]
where \( w_0 = 1 + i + j + 2k \), \( w_1 = 1 + i + 2j + 4k \), \( w_2 = 1 + 2i + 4j + 5k \), and \( w_3 = 2 + 4i + 5j + 9k \).

**Theorem 2.1.** The generating function of the quadrapell quaternion sequence is
\[
W(x) = \frac{w_0 + w_1 x + (w_2 - w_0)x^2 + (w_3 - w_1 - 2w_0)x^3}{1 - x^2 - 2x^3 - x^4}.
\]

**Proof.** Let \( W(x) = \sum_{n=0}^{\infty} w_n x^n \). Then we have
\[
W(x) - x^3 W(x) - 2x^2 W(x) - x W(x) = w_0 + w_1 x + (w_2 - w_0)x^2 + (w_3 - w_1 - 2w_0)x^3 + \sum_{n=4}^{\infty} (w_n - w_{n-2} - 2w_{n-3} - w_{n-4})x^n.
\]
Since, for each \( n \geq 4 \), the coefficient of \( x^n \) is zero in the right-hand side of this equation, we easily get
\[
W(x) = \frac{w_0 + w_1 x + (w_2 - w_0)x^2 + (w_3 - w_1 - 2w_0)x^3}{1 - x^2 - 2x^3 - x^4}.
\]

**Theorem 2.2.** For \( n \geq 0 \), we have
\[
w_n = A \alpha^n - \beta^n + B \beta^{n+1} - \alpha^{n+1} + C \gamma^n - \delta^n + D \gamma^{n+1} - \delta^{n+1}
\]
where
\[
A = \frac{-1 + 2i + j + 3k}{2}, \quad B = \frac{2 + i + 3j + 4k}{2}, \quad C = \frac{1 - j + k}{2}, \quad D = \frac{i - j}{2}.
\]

**Proof.** By using the partial fraction decomposition, we obtain
\[
W(x) = \frac{2 + i + 3j + 4k}{1 - x - x^2} + \frac{(1 - j + k)x}{1 + x + x^2} + \sum_{n=0}^{\infty} \left( A \alpha^n - \beta^n + B \beta^{n+1} - \alpha^{n+1} + C \gamma^n - \delta^n + D \gamma^{n+1} - \delta^{n+1} \right) x^n.
\]
By the equality of the generating function we have
\[
w_n = A \alpha^n - \beta^n + B \beta^{n+1} - \alpha^{n+1} + C \gamma^n - \delta^n + D \gamma^{n+1} - \delta^{n+1}.
\]
Now we give some summation formulas for the quadrapell quaternions. We only prove the first claim. The others could be proven similarly.

**Theorem 2.3.** For \( k \geq 0 \), we have
1. \( \sum_{n=0}^{k} w_n = w_{k+2} - (i + 3j + 3k), \)
2. \( \sum_{n=0}^{k} \alpha^n = w_{k+2} - (2i + 2j + 3k), \)
3. \( \sum_{n=0}^{k} \beta^n = w_{k+2} - (i + 2j + 4k), \)
4. \( \sum_{n=0}^{k} \gamma^n = w_{k+2} - (j + 2k), \)
5. \( \sum_{n=0}^{k} \delta^n = w_{k+2} - (-1 + i + j + k). \)

**Proof.** From the definition of quadrapell quaternions, we have
\[
\sum_{n=0}^{k} w_n = \sum_{n=0}^{k} (D_n + D_{n+1} + D_{n+2} + D_{n+3}) k
\]
\[
= \sum_{n=0}^{k} D_n + \sum_{n=0}^{k} D_{n+1} - \sum_{n=0}^{k} D_{n+2} - \sum_{n=0}^{k} D_{n+3} k
\]
If we use the Theorem 2.5 and Theorem 2.6 in [7], we get
\[
\sum_{n=0}^{k} w_n = \sum_{n=0}^{k} D_n + \sum_{n=0}^{k} D_{n+1} - \sum_{n=0}^{k} D_{n+2} - \sum_{n=0}^{k} D_{n+3} k
\]
\[
= w_{k+2} - (i + 3j + 3k).
\]
3. Binomial Transform of Quadrapell Quaternions

In this section, we give the binomial transform of quadrapell quaternion sequence and obtain some certain identities related to this binomial transform.

Definition 3.1. Let $w_n$ be the $n$th quadrapell quaternion. Then the binomial transform of quadrapell quaternion is

$$b_n = \sum_{i=0}^{n} \binom{n}{i} w_i.$$ 

From the above definition, it is clear that

$$b_{n+1} = \sum_{i=0}^{n} \binom{n}{i} (w_i + w_{i+1})$$

and

$$b_{n+1} = b_n + \sum_{i=0}^{n} \binom{n}{i} w_{i+1}.$$ 

Also, we can see that for even $n$,

$$b_n = w_{2n}.$$

Theorem 3.1. For $n \geq 0$, \{b_n\} satisfies the following recurrence relation:

$$b_{n+1} = 4b_{n+3} - 5b_{n+2} + 4b_{n+1} - b_n,$$

where $b_0 = 1 + i + j + 2k$, $b_1 = 2 + 2i + 3j + 6k$, $b_2 = 4 + 5i + 9j + 15k$, and $b_3 = 9 + 14i + 24j + 38k$.

Proof. Firstly, consider the following recurrence relation:

$$b_{n+1} = x b_{n+3} + y b_{n+2} + z b_{n+1} + t b_n.$$

If we find the equivalents of the $b_i$, $b_n$, $b_{n+1}$ and $b_{n+2}$ from the above equation and solve the system by Cramer rule, we get $x = 4$, $y = -5$, $z = 4$, and $t = -1$.

Theorem 3.2. The generating function of the sequence \{b_n\} is

$$b(x) = \frac{w_0 + (-3w_0 + w_1)x + (w_2 - 2w_1 + 2w_0)x^2 + (w_3 - w_2 - 2w_1)x^3}{1 - 4x - 5x^2 - 4x^3 + x^4}.$$ 

Proof. If we use the transformation given by Gould in [1], we obtain

$$b(x) = \frac{1}{1-x} W\left(\frac{x}{1-x}\right)$$

as desired.

Lastly, we give a Binet-like formula for the sequence \{b_n\}.

Theorem 3.3. For $n \geq 0$, we have

$$b_n = E(-1)^n \gamma^{2n} - \beta^{2n} + F(-1)^{n+1} \gamma^{2n+1} - \beta^{2n+1} + G \frac{\alpha^{3n} - \beta^{3n}}{\alpha - \beta} + H \frac{\alpha^{2n+2} - \beta^{2n+2}}{\alpha - \beta},$$

where $E = \frac{1 - i + k}{2}$, $F = \frac{i - j}{2}$, $G = \frac{-3 + i - 2j - k}{2}$, $H = \frac{2 + i + 3j + 4k}{2}$.

Proof. Recall that the generating function of the sequence \{b_n\} is

$$b(x) = \frac{w_0 + (-3w_0 + w_1)x + (w_2 - 2w_1 + 2w_0)x^2 + (w_3 - w_2 - 2w_1)x^3}{1 - 4x - 5x^2 - 4x^3 + x^4}.$$
Thus, we obtain
\[
b(x) = \left(\frac{1-i+k}{2} \right) x + \frac{i-j}{2} + \left(\frac{-3+i-2j-k}{2} \right) x + \frac{2+i+3j+4k}{2}
\]
\[
\cdot \frac{1-x+x^2}{1-3x+x^2}
\]
\[
= \sum_{n=0}^{\infty} \left[ E(-1)^n \frac{\gamma^{2n} - \delta^{2n}}{\gamma - \delta} + F(-1)^{n+1} \frac{\gamma^{2n+2} - \delta^{2n+2}}{\gamma - \delta} + G \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} + H \frac{\alpha^{2n+2} - \beta^{2n+2}}{\alpha - \beta} \right] x^n.
\]
By the equality of the generating function we have
\[
b_n = E(-1)^n \gamma^{2n} - \delta^{2n} + F(-1)^{n+1} \gamma^{2n+2} - \delta^{2n+2} + G \alpha^{2n} - \beta^{2n} + H \frac{\alpha^{2n+2} - \beta^{2n+2}}{\alpha - \beta}.
\]

4. References