



Forced Vibration Analysis of Non-Uniform Piezoelectric Rod by the Complementary Functions Method

Değişken Kesitli Piezoelektrik Çubuğun Tamamlayıcı Fonksiyonlar Yöntemi ile Zorlanmış Titreşim Analizi

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Abstract

Piezoelectric materials, which have fast response and low energy usage features, are widely used in sensors and actuators. Due to the active role of their working principle, it is important to know the vibration characteristic of each piezoelectric material. In this paper, forced vibration analysis of arbitrary non-uniform piezoelectric rod has been performed. The governing differential equations have variable coefficients which are functions of mechanical and electrostatic properties. Analytical solution of these linear differential equations is limited to specific cross-section area models, so numerical method is inevitable. Numerical model of the forced vibration of cantilever piezoelectric (PZT-4) rod with an arbitrary non-uniform cross-section area is obtained in the Laplace space and then solved numerically by the Complementary Functions Method (CFM). Solutions were transformed from the Laplace domain to the time domain by applying modified Durbin's procedure. The technique is validated for a uniform piezoelectric rod that can also be solved analytically. In order to demonstrate the effect of arbitrary geometry on the dynamic feature of the rod, numerical examples are employed.

Keywords: Complementary functions method, Durbin method, Forced vibration, Laplace transform, Piezoelectric rod

Öz

Sahip oldukları hızlı tepki verme ve düşük enerji gereksinimi gibi özellikleri sayesinde piezoelektrik özelliğe sahip malzemeler algılayıcı ve uyarıcı olarak sıklıkla kullanılmaktadır. Çalışma prensiplerinde titreşim özellikleri etkin rol oynadığından, piezoelektrik malzemelerin titreşim karakteristiklerini bilmek önemlidir. Bu çalışmada kesiti keyfi olarak değişen piezoelektrik bir çubuğun zorlanmış titreşim analizi yapılmıştır. Problemi ifade eden diferansiyel denklem, mekanik ve elektrostatik özelliklerin fonksiyonu olan değişken katsayılara sahiptir. Bu tür bir lineer diferansiyel denklemin analitik çözümü bazı özel kesit alanları ile sınırlı olduğundan sayısal çözümlere kaçılmazdır. Keyfi kesit alanına sahip konsol piezoelektrik çubuğun (PZT-4) zorlanmış titreşimi için matematiksel model Laplace uzayında elde edilmiş ve Tamamlayıcı Fonksiyonlar Yöntemi ile sayısal olarak çözülmüştür. Çözümlerin zaman uzayına dönüşümü modifiye Durbin yöntemi uygulanarak gerçekleştirilmiştir. Çözüm yöntemi, analitik olarak da çözülebilen üniform piezoelektrik çubuk için de geçerlidir. Keyfi kesit değişiminin çubuğun dinamik davranışına etkisini göstermek için sayısal örnekler verilmiştir.

Anahtar Kelimeler: Tamamlayıcı fonksiyonlar yöntemi, Durbin yöntemi, Zorlanmış titreşim, Laplace dönüşümü, Piezoelektrik çubuk

1. Introduction

Smart materials are defined as materials whose geometrical and structural characteristics can be changed in accordance

with the nature of their expected duty. Piezoelectric materials that belong to the smart materials class are able to deform under an electric field or produce an electric signal as a result of any mechanical effect. Electromechanically polarized piezoelectric materials are frequently used as a transducer, sensor and actuator due to their unique features including simple structure, nonflammable, electromagnetic noise free, low energy usage and high operation frequency.

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Piezoelectric transducers can be considered as a rod in terms of mathematical modeling. Thus, it is important to know the vibration behavior of piezoelectric rods with different cross-section area models. Some structural elements, such as rods and beams are designed with variable cross-section due to technical advantages in some applications. Vibration behavior of nonuniform rods with variable cross-section was investigated extensively in the literature (Eisenberg 1991-Yardımoğlu, Aydın 2011 and Celebi et al. 2011). Studies on the piezoelectricity are usually related to finite and infinite dimensional structures in different geometries such as thin rods, solid or hollow cylinders, plates, discs, cylindrical shells. When geometry and material properties get more complicated, applying numerical methods would become inevitable. Chen and Zhang (2009) obtained an exact solution of a nonuniform cross-section rod. One dimensional equation of piezoelectromagnetic rectangular beam for the flexure and extension with shear deformation are derived by Zhang et al. (2009). They investigated the magnetoelectric effects in fiber for both piezoelectric and piezomagnetic composites and compared magnetoelectric effects in fibers and in thin films. It is concluded that magnetoelectric effects in fibers are considerably stronger than those in thin films. Free vibration analysis of the piezoelectric transformer is performed by an accurate electromechanical model with Hamilton's principle by Nadal and Pigache (2009). Yang and Zhife (2009) used elasticity theory with state-space based differential quadrature method to analyze the free vibrations of a functionally graded piezoelectric beam for different boundary conditions. The natural frequency and static behaviour of a flexensional actuator are developed with the help of Hamilton's principle and solved with use of the perturbation method by Przybylski (2015). Exact closed-form solution of a piezoelectric gyroscope for free and forced vibrations are obtained by Yand and Fang (2003). Airy stress function method is employed to get the analytical solution of a functionally graded piezoelectric cantilever beam by Shi and Chen (2004). Free longitudinal vibration of non-uniform piezoelectric rod is solved numerically with the CFM and pseudospectral Chebyshev method by Eker et al. (2015) and Yarımpabuç et al. (2016).

In this study, forced vibration of cantilever piezoelectric (PZT-4) rod with arbitrary non-uniform cross-section area is solved in the Laplace domain numerically by the CFM under four different dynamic load functions. The CFM, theoretically explained in the literature by Aktaş (1972), Agarwal (1982), Roberts and Shipman (1979), Calim (2016)

and Tutuncu and Temel (2009, 2013) is infused into analysis to transform the boundary value problem to a system of an initial-value problem, which can be easily solved by a fifth order Runge-Kutta method (RK5) with great accuracy (Chapra 1998). Durbin's Laplace inversion procedure that has been evinced to be an efficient and accurate inversion method is used to get the results in the time domain (Temel et al. 2014). The numerical and analytical results for a uniform piezoelectric rod are compared in order to show the accuracy of the method. After then, numerical examples are employed to demonstrate the influence of arbitrary geometry on the dynamic feature of the rod.

2. Material and Methods

Rosen type transducers provide the most efficient usage in applications. They consist of two parts, a driving portion and receiving portion, and operate by making use of extensional vibrations of these parts. Each of the driving portion and the receiving portion with different coordinates and under different polarization conditions can be considered as a piezoelectric rod. Consider a non-uniform piezoelectric rod polarized along the longitudinal axis shown in Figure 1.

Constitutive equations of piezoelectric materials that exhibit linear behavior define electromechanical properties and can be derived in various ways to highlight desired properties (Chen 2009). Under the assumption of material properties do not change along the x-axis, and with the consideration of mechanical and electrostatic equations together, the governing equation of the system can be written in the following form (Chen 2009, Eker 2015).

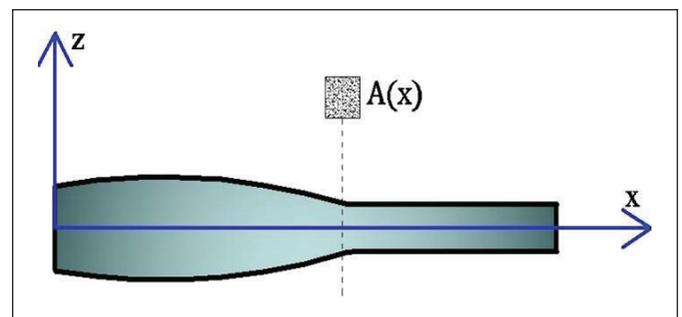


Figure 1. Piezoelectric rod with non-uniform cross-section.

$$\frac{d^2 u}{dx^2} + \frac{1}{A(x)} \frac{dA(x)}{dx} \frac{du}{dx} = \frac{\rho}{\bar{c}_{11}} \frac{\partial^2 u}{\partial t^2} \quad (1)$$

where u is the extensional displacement, $A(x)$ is the cross-sectional area varying along the x -axis, ρ is the mass density, $\bar{c}_{11} = \bar{c}_{11} + \frac{\bar{e}_{11}^2}{\bar{\epsilon}_{11}}$; \bar{c}_{11} , \bar{e}_{11} and $\bar{\epsilon}_{11}$ are elastic, piezoelectric and

dielectric constants for longitudinal motion, respectively. Clamped-free supported rod which is electrically open on two ends is considered, therefore the boundary conditions become,

$$u(0,t) = 0, \quad \frac{\partial u}{\partial x}(l,t) = \frac{P(t)}{\bar{c}_{11}A(l)}$$

where $P(t)$ represents the load functions (Figure 2) applied to the end of the rod (Temel et al., 2014). Taking Laplace transform of Equation (1) with zero initial conditions $(u|_{t=0} = \frac{\partial u}{\partial t}|_{t=0} = 0)$ gives

$$\bar{U}'' + N(x,s)\bar{U}' + Q(x,s)\bar{U} = 0 \tag{3}$$

with

$$N(x,s) = \frac{1}{A(x)} \frac{dA(x)}{dx} \quad \text{and} \quad Q(x,s) = -\frac{s^2 \rho}{\bar{c}_{11}} \tag{4}$$

where ()' represents the derivative with regard to x , $\bar{U} = U(x,s)$ is the Laplace transformation of u and s is the Laplace parameter.

Under the effect of four different load functions on the free end ($x = l$), the boundary conditions become

$$U(0,s) = 0 \quad \text{and} \quad \frac{\partial U}{\partial x}(l,s) = \frac{p(s)}{\bar{c}_{11}A(l)} \tag{5}$$

where $p(s)$ is the Laplace transform of the load $P(t)$.

2.1. Complementary Functions Method

The CFM which is used to solve forced vibration analysis of piezoelectric rod is based on the transformation of the solutions of two-point boundary values problems (BVP) to an initial-value problem (IVP). A simple solution schedule can be obtained by the method even if the rod has various cross sections and applied force conditions. General closed-form solution of the linear governing differential equation that given above cannot be obtained except for simple cross sections. The complete solution of Equation (3) is

$$\bar{U} = b_j \bar{U}_j, \quad j = 1,2 \tag{6}$$

where \bar{U}_j is the linearly independent homogeneous solution in the Laplace domain. The coefficients b_j are determined via the boundary conditions. The CFM is started by assuming $\bar{U}_1 = Y_1^{(i)}$ and $\bar{U}_2 = Y_2^{(i)}$ (Tutuncu and Temel 2009, 2013), which means

$$(Y_1^{(i)})' = Y_2^{(i)} \tag{7}$$

$$(Y_2^{(i)})' = -N(x,s)Y_2^{(i)} - Q(x,s)Y_1^{(i)} \tag{8}$$

To get each homogeneous solution, the system of equation (Equation (7-8)) can be solved numerically. In order to ensure linear independency, the Kronecker delta is used as initial conditions (Roberts 1979),

$$Y_j^{(i)} = \delta_{ji}, \quad j = 1,2. \tag{9}$$

Equations (7) and (8) constitute a system of equations for the homogeneous solution along with the initial conditions. The RK5 will be used for all cases considered. With this solution procedure, $U(x,s)$ and its first derivative are calculated together simultaneously. Imposing the boundary conditions to the present problem leads to a system of algebraic equations for the coefficients b_1 and b_2 as follows:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} RHS1 \\ RHS2 \end{bmatrix} \tag{10}$$

Here, A_{ij} includes the values of the homogeneous solutions at the boundary points. $RHS1$ and $RHS2$ contain values of the boundary conditions that include external load subjected to the rod. These points will be illustrated in the following sections.

2.2. Formulation

Three different cross-section forms (Figure(3)) are considered in the present study. The first two cross-section variations for some simple form can be solved analytically, however, the third one can not.

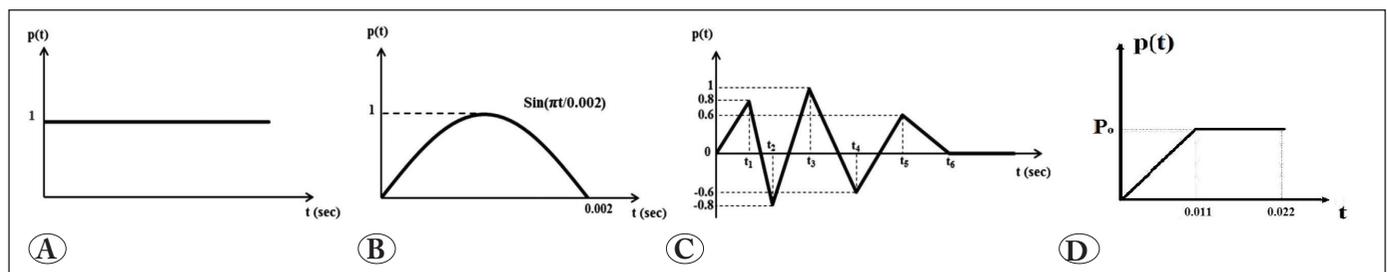


Figure 2. Applied load functions: **A)** Step load function, **B)** Sinusoidal impulsive load function, **C)** Arbitrary load function $t_1 = 0.0014$, $t_2 = 0.0024$, $t_3 = 0.004$, $t_4 = 0.005$, $t_5 = 0.006$, $t_6 = 0.007$, **D)** Ramped Load function ($b = 0.011$).

2.2.1. Power-Form Variation

Consider that the cross-sectional area varying with the following power-form,

$$A(x) = A_0 \left(a + b \frac{x}{l} \right)^n \tag{11}$$

where a and b are dimensionless parameters and n represents inhomogeneity parameters. Substituting Equation (11) into Equation (3) turns into linear ordinary differential equations,

$$\bar{U}'' + N(x,s)\bar{U}' + Q(x,s)\bar{U} = 0 \tag{12}$$

with

$$N(x,s) = \frac{nb}{(al+bx)} \text{ and } Q(x,s) = -\frac{s^2 p}{\bar{c}_{11}} \tag{13}$$

Letting $\bar{U}_i = Y_1^{(i)}$ and $\bar{U}'_i = Y_2^{(i)}$ yields

$$(Y_1^{(i)})' = Y_2^{(i)} \tag{14}$$

$$(Y_2^{(i)})' = -N(x,s)Y_2^{(i)} - Q(x,s)Y_1^{(i)}. \tag{15}$$

Following the steps outlined in Section 2.1, the constants b_j can now be found from the system given by Equation (10) where,

$$\begin{aligned} A_{11} &= U_1(0,s) & A_{12} &= U_2(0,s) \\ A_{21} &= \bar{U}'_1(l,s) & A_{22} &= \bar{U}'_2(l,s) \\ RHS1 &= 0 & RHS2 &= \frac{p(s)}{\bar{c}_{11}A(l)} \end{aligned} \tag{16}$$

Then, the displacement and first derivative can be obtained at the collocation points.

2.2.2 Exponential-Form Variation

Consider that the cross-sectional area varying with the following exponential-form,

$$A(x) = A_0 e^{-\frac{nx}{l}} \tag{17}$$

where n represents the inhomogeneity parameters. Substituting Equation (17) into Equation (3) turns into linear ordinary differential equations,

$$\bar{U}'' + N(x,s)\bar{U}' + Q(x,s)\bar{U} = 0 \tag{18}$$

with

$$N(x,s) = \frac{n}{l} \text{ and } Q(x,s) = -\frac{s^2 \rho}{\bar{c}_{11}} \tag{19}$$

Letting $\bar{U}_i = Y_1^{(i)}$ and $\bar{U}'_i = Y_2^{(i)}$ yields

$$(Y_1^{(i)})' = Y_2^{(i)} \tag{20}$$

$$(Y_2^{(i)})' = -N(x,s)Y_2^{(i)} - Q(x,s)Y_1^{(i)} \tag{21}$$

This equation system is solved by a similar procedure that was mentioned previously in Section 2.1 by using Equation (10) where,

$$\begin{aligned} A_{11} &= U_1(0,s) & A_{12} &= U_2(0,s) \\ A_{21} &= \bar{U}'_1(l,s) & A_{22} &= \bar{U}'_2(l,s) \\ RHS1 &= 0 & RHS2 &= \frac{p(s)}{\bar{c}_{11}A(l)} \end{aligned} \tag{22}$$

2.2.3 Cosinusoidal-Form Variation

Consider that the cross-sectional area varying with the following cosinusoidal-form,

$$A(x) = A_0 \text{Cos}^2 \left[a \frac{nx}{l} + b \right] \tag{23}$$

where n represents inhomogeneity parameters. Substituting Equation (17) into Equation (3) yields,

$$\bar{U}'' + N(x,s)\bar{U}' + Q(x,s)\bar{U} = 0 \tag{24}$$

with

$$N(x,s) = -\frac{2na}{l} \text{Tan} \left[a \frac{nx}{l} + b \right] \text{ and } Q(x,s) = -\frac{s^2 \rho}{\bar{c}_{11}} \tag{25}$$

Letting $\bar{U}_i = Y_1^{(i)}$ and $\bar{U}'_i = Y_2^{(i)}$ yields

$$(Y_1^{(i)})' = Y_2^{(i)} \tag{26}$$

$$(Y_2^{(i)})' = -N(x,s)Y_2^{(i)} - Q(x,s)Y_1^{(i)} \tag{27}$$

This equation system is also solved by a similar procedure that was mentioned in Section 2.1 by using Equation (10) where,

$$\begin{aligned} A_{11} &= U_1(0,s) & A_{12} &= U_2(0,s) \\ A_{21} &= \bar{U}'_1(l,s) & A_{22} &= \bar{U}'_2(l,s) \\ RHS1 &= 0 & RHS2 &= \frac{p(s)}{\bar{c}_{11}A(l)} \end{aligned} \tag{28}$$

3. Results

In this research, a general objective computer program is coded in Matlab to analyze the forced vibration of arbitrary non-uniform piezoelectric rods. Material constants for PZT-4 was taken from (Ding and Chen, 2001). In the solution procedure of the initial value problem based on the CFM, RK5 algorithm is used. Inverse Laplace transformation into the time domain is taken by modified Durbin's method. In modified Durbin's procedure step size and Laplace parameters were chosen as 8.5937×10^{-5} and 128 respectively. Four types of dynamic axial end force are used in the analyses which are given in Figure 2; step load, sinusoidal impulsive load, arbitrary load and periodic load. The geometrical models for different cases are given in Section 2. The dimensionless parameters " a , b " are taken as 0.8, -0.2 and the inhomogeneity parameter " n " is taken as 1, 1.5 and 1.8 for all cases considered. The results are correspond to non-uniform cross-section with constant material properties.

In order to ensure the efficiency and the adequacy of the present method, analytical and the CFM results for a uniform rod are presented in Figure 4. It can be noted from the figure that the CFM results match quite well with the analytical results. Observing these results disclosed the great accuracy and efficiency accomplished by the CFM, and the calculations executed at only 11 points through the length yielded six-digit accuracy. Figure 5 (a-k) show that results of the displacement at the end of the rod ($x = l$) for different

geometrical models subjected to dynamic loads. As shown in Figure 5 (a-k), the inhomogeneity parameter is a useful parameter for controlling the displacement amplitude.

4. Conclusion

In this work, forced vibration analysis of the arbitrary non-uniform piezoelectric rod is presented. The cross-sectional area of the rods is varying continuously in the axial direction. Under the Laplace transformation, the partial differential

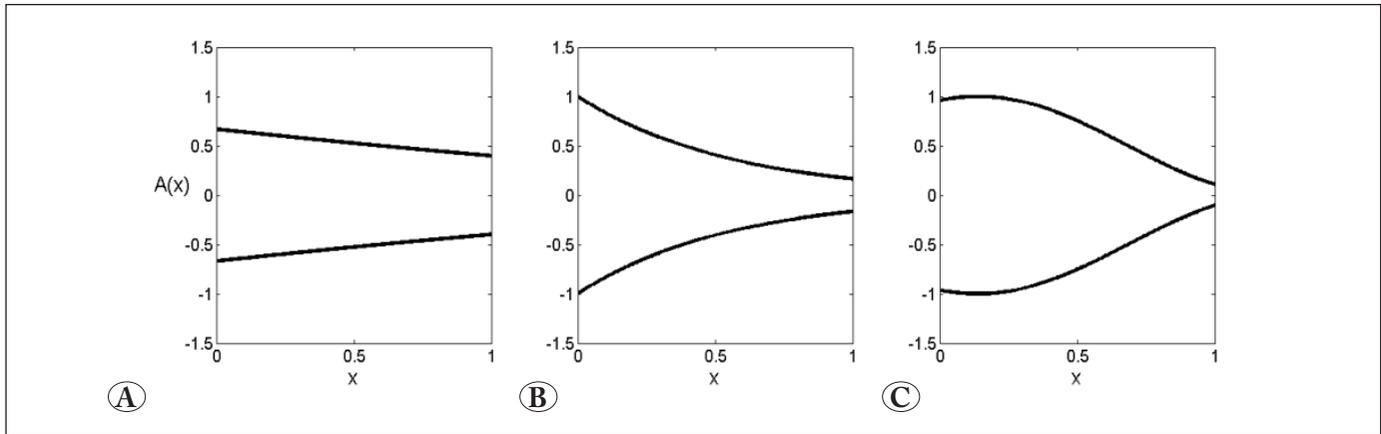


Figure 3. Considered geometry of the rod: **A)** Power form cross-section, **B)** Exponential form cross-section, **C)** Cosinusoidal form cross-section.

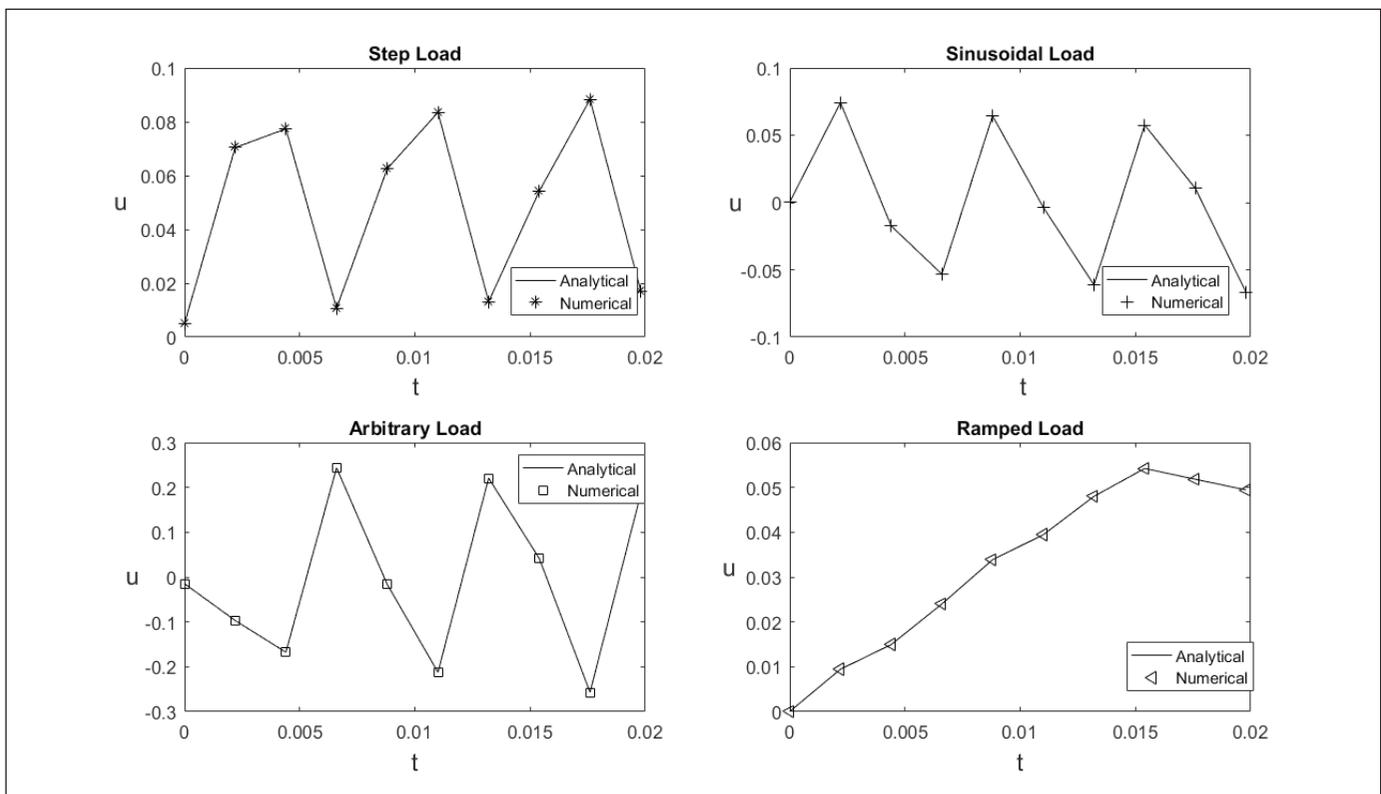


Figure 4. Comparison of displacement (u) at the end of the rod ($x = l$) for constant cross-section ($u \times 10^{-9}$)

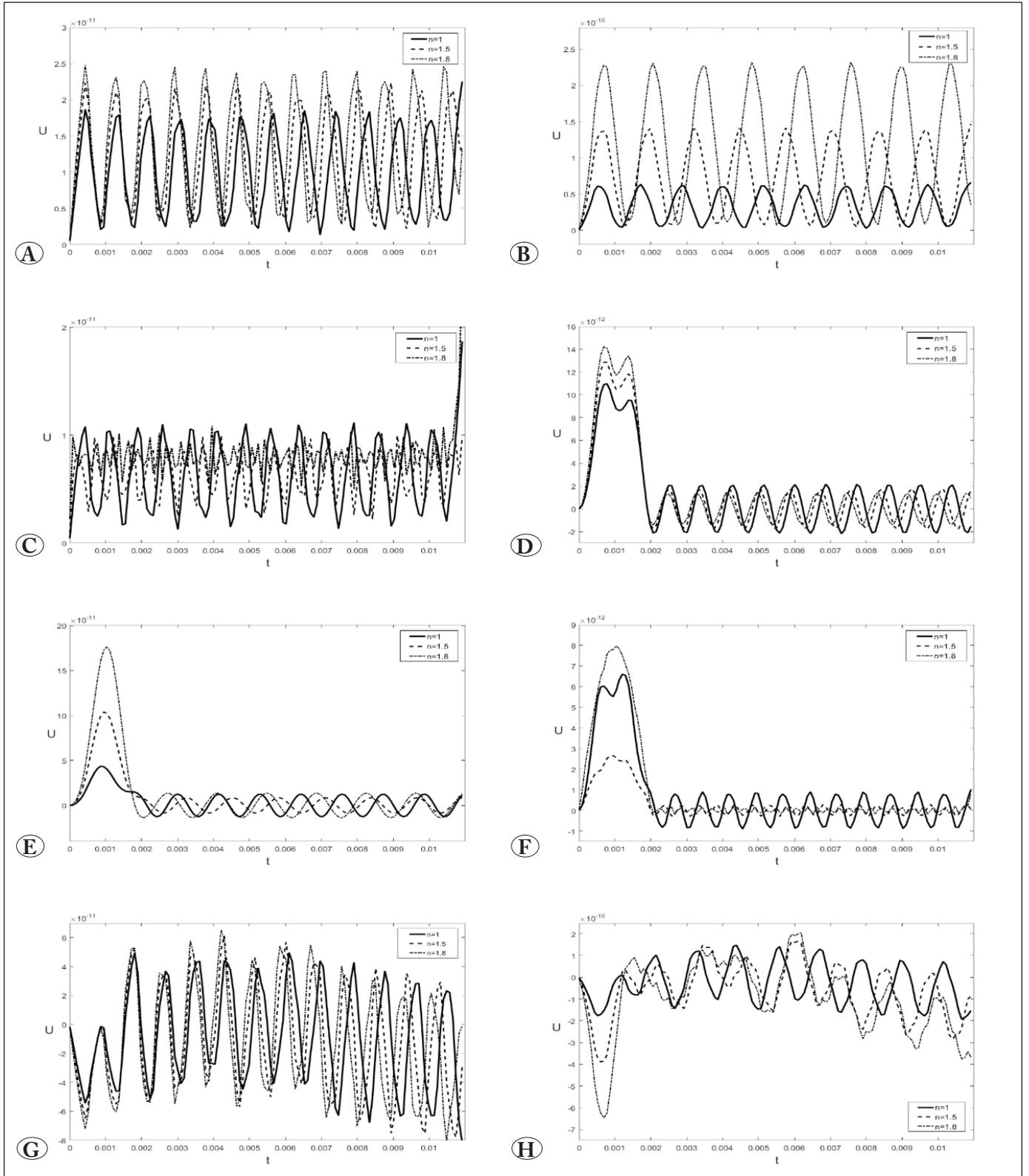


Figure 5. Forced vibration analysis of rod at different cross-sectional areas under different types of load ($a=0.8$, $b=-0.2$, $A_0=1$). **A)** Power form cross section under step load. **B)** Exponential form cross section under step load. **C)** Cosinusoidal cross section under step load. **D)** Power form cross section under sinusoidal load. **E)** Exponential form cross section under sinusoidal load. **F)** Cosinusoidal form cross section under sinusoidal load. **G)** Power form cross section under arbitrary load. **H)** Exponential form cross section under arbitrary load.

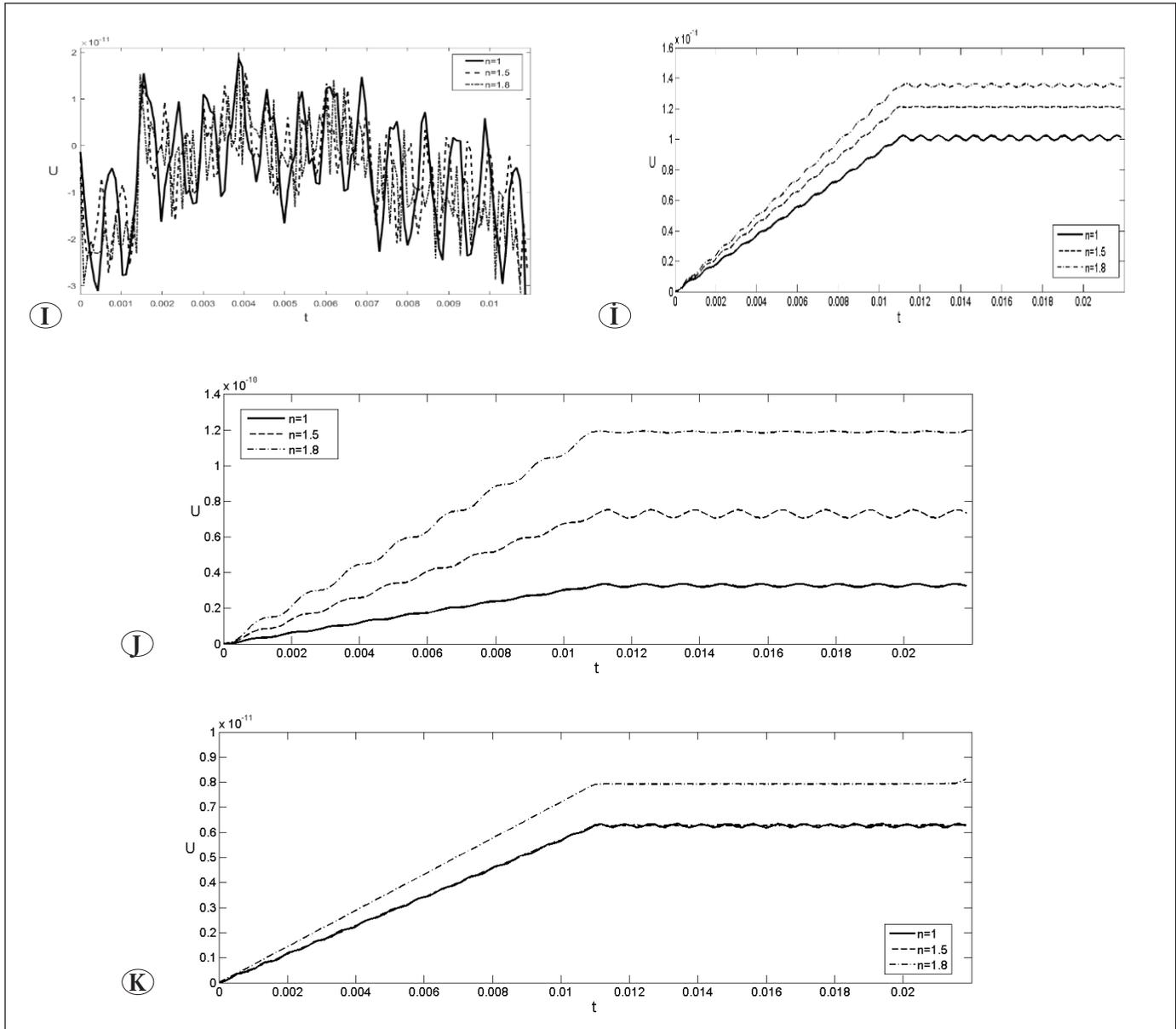


Figure 5. **I)** Cosinusoidal form cross section under arbitrary load. **I)** Power form cross section under ramped load. **J)** Exponential form cross section under ramped load. **K)** Cosinusoidal form cross section under ramped load.

equation is transformed into a time-independent boundary-value problem in spatial domain which is computed by the CFM. The CFM method converts the problem to a system of initial-value problem that can be solved by any conventional method in the literature. The system of the initial value problem is solved by the RK5. Inverse transformation of the results into the time-space is taken by modified Durbin's method. From the results presented above the following conclusions are reached:

- The solution strategy provided here presents a viable method to solve cross-sectional area problems with

high accuracy. It is also noted that the solution requires little to no computational costs and is straightforward.

- The approach further introduces the elimination of separate computations for mode shapes and natural frequencies. Obtaining the forced vibration response values are achieved directly.
- In terms of design perspective, the inhomogeneity parameter constitutes an adjustment variable for particular applications. This further enables control on the displacement amplitude.

Appendix A: The Modified Durbin's Inverse Laplace Transform

Using a numerical inverse Laplace transform technique is inevitable to get the values in the time domain (Celebi et al. 2011). For this, the fast Fourier transform based modified Durbin's inverse Laplace transform technique is used (Durbin 1974s). Modified Durbin's formulation for inverse Laplace transform is outlined below:

$$f(t_j) \cong \frac{2e^{aj\Delta t}}{T} \left[-\frac{1}{2} \text{Re}\{\bar{F}(a)\} + \text{Re}\left\{ \sum_{m=0}^{M-1} (A(m) + iB(m))e^{(i\frac{2\pi}{M})^m} \right\} \right], (j = 0, 1, 2, \dots, M-1) \tag{29}$$

where

$$A(m) = \sum_{l=0}^L \text{Re}\left\{ \bar{F}(a + i(m + lM)) \frac{2\pi}{T} \right\} \tag{30}$$

$$B(m) = \sum_{l=0}^L \text{Im}\left\{ \bar{F}(a + i(m + lM)) \frac{2\pi}{T} \right\} \tag{31}$$

Here, i is the complex number, $s_m = a + i\frac{2\pi m}{T}$ is the m^{th} Laplace parameter. There are M units of equal time intervals and T is the solution interval. $f(t)$ is calculated for all $t_j = j\Delta t = j\frac{T}{N}$, ($j = 0, 1, 2, \dots, M-1$). The most suitable way for this transform is using the $5 \leq aT \leq 10$ interval. For numerical examples in this study, the value of aT is chosen as 7.5. Eventually, results can be modified multiplying each term by Lanczos (L_m) factors due to amend results in the Laplace domain (Celebi et al. 2011).

$$f(t_j) \cong \frac{2e^{aj\Delta t}}{T} \left[-\frac{1}{2} \text{Re}\{\bar{F}(a)\} + \text{Re}\left\{ \sum_{m=0}^{M-1} \{\bar{F}(s_m)L_m\}e^{(i\frac{2\pi}{M})^m} \right\} \right] \tag{32}$$

$$L_m = \begin{cases} L_0 = 1, & \text{for } m = 0 \\ L_m = \frac{\sin \frac{m\pi}{M}}{\frac{m\pi}{M}}, & \text{for } m > 0 \end{cases} \tag{33}$$

Appendix B: Laplace Transformations of Considered Loads

Step Load: $\mathcal{L}[P(t)] = \frac{1}{s}$

Sinusoidal Impulsive Load: $\mathcal{L}[P(t)] = \frac{500\pi}{s^2 + (500\pi)^2} + e^{-0.002s} \left(\frac{500\pi}{s^2 + (500\pi)^2} \right)$

$$\text{Arbitrary Load: } P(t) = \begin{cases} \frac{4000t}{7} \dots\dots\dots 0 \leq t \leq t_1 \\ 3.04 - 1600t \dots\dots\dots t_1 \leq t \leq t_2 \\ 1125t - 3.5 \dots\dots\dots t_2 \leq t \leq t_3 \\ 7.4 - 1600t \dots\dots\dots t_3 \leq t \leq t_4 \\ 1200t - 6.6 \dots\dots\dots t_4 \leq t \leq t_5 \\ 4.2 - 600t \dots\dots\dots t_5 \leq t \leq t_6 \\ 0 \dots\dots\dots t > t_6 \end{cases}$$

$$\begin{aligned} \mathcal{L}[P(t)] = & \frac{4000}{7s^2} + e^{-t_1s} \left[\frac{3.04}{s} - \frac{2171.4286}{s^2} - \frac{2171.4286t_1}{s} \right] \\ & + e^{-t_2s} \left[\frac{-6.54}{s} + \frac{2725}{s^2} + \frac{2725t_2}{s} \right] \\ & + e^{-t_3s} \left[\frac{10.9}{s} - \frac{2725}{s^2} - \frac{2725t_3}{s} \right] \\ & + e^{-t_4s} \left[\frac{-14}{s} + \frac{2800}{s^2} + \frac{2800t_4}{s} \right] \\ & + e^{-t_5s} \left[\frac{10.8}{s} - \frac{1800}{s^2} - \frac{1800t_5}{s} \right] \\ & + e^{-t_6s} \left[\frac{-3}{s} + \frac{300}{s^2} + \frac{300t_6}{s} \right] \end{aligned}$$

Ramped load: $\mathcal{L}[P(t)] = \frac{e^{-0.011s} - e^{-0.022s} + 0.011s}{0.011s^2 - 0.011s^2 e^{0.022s}}$

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