On Some Congruences with the Terms of Second Order Sequence and Harmonic Numbers

İkinci Mertebeden Dizinin Terimlerini ve Harmonik Sayıları İçeren Bazı Kongrüanslar

Neşe Ömür, Sibel Koparal*
Kocaeli University, Faculty of Arts and Sciences, Department of Mathematics, Kocaeli, Turkey

Abstract

In this paper, we give the generalization of the congruences in (1.2) and (1.3). For example, for \( b \in \mathbb{Z}\setminus\{0\} \), we have
\[
\sum_{k=0}^{n-1} \frac{U_{n+k}(1,b)}{b^k} H_k \equiv 0 \pmod{p}
\]
where \( p \) is a prime such that \( p \nmid \Delta \), \( \Delta = 1-4b^2 \) and \( \varepsilon = \left(1 - \left(\frac{\Delta}{p}\right)\right)/2 \)

Keywords: Congruence, Harmonic numbers, The second order sequences

1. Introduction

The second order sequences \( \{U_n(A,B)\} \) and \( \{V_n(A,B)\} \) are defined for \( n>0 \) by

\[
U_n(A,B) = AU_n(A,B) - BU_{n-1}(A,B)
\]
and

\[
V_n(A,B) = AV_n(A,B) - BV_{n-1}(A,B)
\]
in which \( U_n(A,B) = 0, U_1(A,B) = 1 \) and \( V_n(A,B) = 2, V_1(A,B) = A \), respectively, where \( A \) and \( B \) are arbitrary integers.

The Binet formulae of sequences \( \{U_n(A,B)\} \) and \( \{V_n(A,B)\} \) are

\[
U_n(A,B) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n(A,B) = \frac{\alpha^n + \beta^n}{2},
\]
respectively, where \( \alpha, \beta = \left(A \pm \sqrt{A^2 - 4B}\right)/2 \). If \( A = 1 \) and \( B = -1 \), then \( U_1(1,-1) = F_n \) (nth Fibonacci number) and \( V_1(1,-1) = L_n \) (nth Lucas number).

For \( n \in \mathbb{N} \) \( \{1,2,...\} \), harmonic numbers are those rational numbers given by
\[
H_0 = 0, \quad H_n = \sum_{k=1}^{n} \frac{1}{k}.
\]

Wolstenholme proved that if \( p > 3 \) is a prime, then
\[
H_{p-1} \equiv 0 \pmod{p^2}
\]
(Wolstenholme 1862). For an odd prime \( p \) and an integer \( a \), \( \left(\frac{a}{p}\right) \) denotes the Legendre symbol given by
\[
\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p \mid a, \\ 1 & \text{if } a \text{ is a quadratic residue modulo } p, \\ -1 & \text{if } a \text{ is a quadratic nonresidue modulo } p. 
\end{cases}
\]

Sun showed the congruences involving harmonic numbers and Lucas sequences (Sun 2012). For example, let \( p > 3 \) be a prime. For \( A,B \in \mathbb{Z} \) with \( p \nmid A \),

\[
\sum_{k=1}^{p-1} V_k(A,B) kA^k H_k \equiv 0 \pmod{p},
\]

\[
\sum_{k=1}^{p-1} U_k(A,B) kA^k H_k \equiv \frac{2}{p} \sum_{k=1}^{p-1} U_k(A,B) (mod \ p),
\]

and for a prime \( p > 5 \), if \( \left(\frac{p}{15}\right) = 1 \),

\[
\sum_{k=1}^{p-1} \frac{U_k(1,4)}{2^k} H_k \equiv 0 \pmod{p},
\]

(1.2)
\[
\Delta U_r(1,b^r) = (\delta - \gamma)^r U_r(1,b^r) = (\delta - \gamma)(\delta^r - \gamma^r) \\
\equiv (\delta - \gamma)^{r+1} = \Delta^{(r+1)/2}(\mod p).
\]

Hence, for \( p \not\equiv \Delta \), we write
\[
U_r(1,b^2) \equiv \Delta^{(r+1)/2} \equiv \left(\frac{\Delta}{p}\right)(\mod p).
\]

Similarly, we get
\[
V_r(1,b^2) = \delta^r + \gamma^r \equiv (\delta + \gamma)^r = 1(\mod p).
\]

It is clearly given that for any prime number \( p \),
\[
U_r(1,b^2) + V_r(1,b^2) = 2U_{r+1}(1,b^2).
\]

For \( \left(\frac{\Delta}{p}\right) = 1 \) and \( p \not\equiv \Delta \), using recurrence relation of the sequence \( \{U_r(1,b^r)\} \) and (2.2), we have
\[
\frac{b^2 U_{r-1}(1,b^r)}{b^2} = U_r(1,b^r) - U_{r+1}(1,b^2)
\]
\[
= U_r(1,b^r) - \frac{1}{2}(U_r(1,b^r) + V_r(1,b^2))
\]
\[
= \frac{1}{2}(U_r(1,b^r) - V_r(1,b^2)) \equiv \left(\frac{\Delta}{p}\right) - \frac{1}{2} = 0(\mod p)
\]

and by the little Fermat Theorem, we get
\[
V_{r+1}(1,b^2) = 2U_{r+1}(1,b^2) - U_{r+1}(1,b^2)
\]
\[
\equiv 2 \equiv 2b^{r+1}(\mod p).
\]

Since the congruence
\[
(V_{r-1}(1,b^2) - 2b^{r+1})(V_{r+1}(1,b^2) + 2b^{r+1})
\]
\[
= (\delta^{r+1} + \gamma^{r+1})^2 - 4(\delta \gamma)^{r+1}
\]
\[
= (\delta^{r+1} - \gamma^{r+1})^2 = \Delta U_{r+1}(1,b^2) \equiv 0(\mod p^2),
\]
we have \( V_{r+1}(1,b^2) \equiv 2b^{r+1}(\mod p^2) \). Thus
\[
2U_r(1,b^r) = U_{r-1}(1,b^2) + V_{r-1}(1,b^2)
\]
\[
\equiv U_{r-1}(1,b^2) + 2b^{r+1}(\mod p^2).
\]

For \( \left(\frac{\Delta}{p}\right) = -1 \), by (2.2), we have
\[
2U_{r+1}(1,b^2) = U_r(1,b^2) + V_{r+1}(1,b^2) \equiv 0(\mod p)
\]

and with the help of recurrence relation of the sequence \( \{U_r(1,b^r)\} \), the little Fermat Theorem and (2.2), we get
\[
V_{r+1}(1,b^2) = 2U_{r+2}(1,b^2) - U_{r+1}(1,b^2)
\]
\[
= -2b^2 U_r(1,b^2) + U_{r+1}(1,b^2)
\]
\[
\equiv 2b^2 \equiv 2b^{r+1}(\mod p).
\]

Considering the congruence
\[
(V_{r+1}(1,b^2) - 2b^{r+1})(V_{r+1}(1,b^2) + 2b^{r+1})
\]
\[
= (\delta^{r+1} + \gamma^{r+1})^2 - 4(\delta \gamma)^{r+1}
\]
\[
= (\delta^{r+1} - \gamma^{r+1})^2 = \Delta U_{r+1}(1,b^2) \equiv 0(\mod p^2),
\]
we have \( V_{r+1}(1,b^2) \equiv 2b^{r+1}(\mod p^2) \). Hence
\[2b^2U_r(1,b^2) = 2(U_{r+1}(1,b^2) - U_{r+1}(1,b^2))
\]
\[= 2U_{r+1}(1,b^2) - (U_{r+1}(1,b^2) + V_{r+1}(1,b^2))
\]
\[= U_{r+1}(1,b^2) - V_{r+1}(1,b^2)\]
\[\equiv U_{r+1}(1,b^2) - 2b^{r+1}(\mod p^r).\]

Combining (2.3) and (2.4), the proof is completed.

Secondly, we give theorem involving the generalization of the congruences in (1.2) and (1.3).

**Theorem 2.** For \(b \in \mathbb{Z}/\{0\},\)

\[\sum_{k=0}^{r-1} \frac{U_{r+1}(1,b^2)}{b^2} H_k \equiv 0 (\mod p),\]  

(2.5)

where \(p\) is a prime such that \(p \neq b\Delta\) and \(\varepsilon = \left(1 - \left(\frac{A}{p}\right)\right)/2.\)

**Proof.**

With the help of (1.4), we get

\[\sum_{k=0}^{r-1} \frac{U_{r+1}(1,b^2)}{b^2} H_k \equiv 0 (\mod p),\]  

where \(p \neq b\Delta.\)

For \(\varepsilon = 0, 1,\) it is enough to show Theorem 2 that

\[\sum_{k=0}^{r-1} b^{r-1-k} U_{r+1}(1,b^2) \equiv \sum_{k=0}^{r-1} \left(\frac{p-1}{k}\right)(-b)^{r-1-k} U_{r+1}(1,b^2) (\mod p^r).\]

Using Binet formula of the sequence \(\{U_{r+1}(1,b^2)\},\) we write

\[\sum_{k=0}^{r-1} b^{r-1-k} \frac{\delta^{r-1} - \gamma^{r-1} + (b^r)}{\delta - \gamma} \equiv \sum_{k=0}^{r-1} \left(\frac{p-1}{k}\right)(-b)^{r-1-k} \frac{\delta^{r-1} - \gamma^{r-1}}{\delta - \gamma} (\mod p^r).\]

By the sums

\[\sum_{k=0}^{r-1} x^k y^{r-1-k} = \frac{x^r - y^r}{x - y} \quad \text{and} \quad \sum_{k=0}^{r-1} \left(\frac{p-1}{k}\right) x^k y^{r-1-k} = (x + y)^{r-1},\]

we write

\[\frac{1}{\delta - \gamma} \left(\delta^r - \gamma^r - b^r - \gamma^r - b^r\right) \equiv \frac{\delta^r (\delta - b)^{r-1} - \gamma^r (\gamma - b)^{r-1}}{\delta - \gamma} (\mod p^r).\]

(2.6)

It is known that

\[(\delta - b)(\gamma - b) = \delta \gamma - b(\delta + \gamma) + b^2 = 2b^2 - b\]

and

\[\delta^r (\gamma - b)(\delta^r - b^r) - \gamma^r (\delta - b)(\gamma^r - b^r)\]

\[= (\delta - \gamma)(b^{r+1} - bU_{r+1}(1,b^2) + b^rU_{r+1}(1,b^2)).\]

So the congruence in (2.6) can rewritten

\[\frac{b^{r+1} - U_{r+1}(1,b^2) + bU_{r+1}(1,b^2)}{2b - 1} \equiv \frac{\delta^r (\delta - b)^{r-1} - \gamma^r (\gamma - b)^{r-1}}{\delta - \gamma} (\mod p^r).\]

By the equalities \((\delta - b)^r = (1 - 2b)\delta\) and \((\gamma - b)^r = (1 - 2b)\gamma,\) we have

\[\frac{\delta^r (\delta - b)^{r-1} - \gamma^r (\gamma - b)^{r-1}}{\delta - \gamma} = \frac{\delta^r ((1 - 2b)\delta)^{r-1} - \gamma^r ((1 - 2b)\gamma)^{r-1}}{\delta - \gamma} = (1 - 2b)^{r-1} U_{(r-1)/2}(1,b^r).\]

Taking \(A = 1\) and \(B = b^r\) in Lemma 1, we write

\[U_{(r-1)/2}(1,b^r) \equiv 0 (\mod p),\]

(2.7)

\[U_{(r-1)/2}(1,b^r) \equiv \left(\frac{1 - 2b}{p}\right) b^{(r-1)/2} (\mod p).\]

(2.8)

For \(\left(\frac{A}{p}\right) = 1,\) by (2.7) and (2.8), we have

\[U_{(r-1)/2}(1,b^r) \equiv 0 (\mod p),\]

\[V_{(r-1)/2}(1,b^r) = 2U_{(r+1)/2}(1,b^r) - U_{(r-1)/2}(1,b^r) \equiv 2 \left(\frac{1 - 2b}{p}\right) (\mod p).\]

and

\[U_{(r-1)/2}(1,b^r) \equiv U_{(r-1)/2}(1,b^r) V_{(r-1)/2}(1,b^r)\]

\[\equiv 0 (\mod p),\]

\[V_{(r+1)/2}(1,b^r) = 2U_{(r+3)/2}(1,b^r) - U_{(r+1)/2}(1,b^r)\]

\[= U_{(r+1)/2}(1,b^r) - 2b^2 U_{(r-1)/2}(1,b^r)\]

\[\equiv -2b^2 \left(\frac{1 - 2b}{p}\right) (\mod p).\]

and

\[U_{(r+1)/2}(1,b^r) = U_{(r+1)/2}(1,b^r) V_{(r+3)/2}(1,b^r)\]

\[\equiv 0 (\mod p),\]

\[V_{(r+3)/2}(1,b^r) = 2U_{(r+5)/2}(1,b^r) - U_{(r+3)/2}(1,b^r)\]

\[= U_{(r+3)/2}(1,b^r) - 2b^2 U_{(r+1)/2}(1,b^r)\]

\[\equiv -2b^2 \left(\frac{1 - 2b}{p}\right) (\mod p).\]

Thus, the right-hand side of (2.6) is congruent to \(U_{(r-1)/2}(1,b^r)/(2(-b)^r)(\mod p^r).\)

Thus (2.6) is equivalent to the congruence

\[\frac{b^{r+1} - U_{r+1}(1,b^2) + bU_{r+1}(1,b^2)}{2b - 1} \equiv \frac{U_{(r-1)/2}(1,b^r)}{2(-b)^r} (\mod p^r).\]

(2.9)
For \( \frac{A}{p} = 1 \), from (2.1), we have \( \epsilon = 0 \), and (2.9) reduces to the congruence
\[
2(b^{p-1} - U_p(1, b^p) + bU_{p-1}(1, b^p)) \equiv (2b - 1)U_{p-1}(1, b^p) \pmod{p^3}.
\]

For \( \frac{A}{p} = -1 \), from (2.1), we have \( \epsilon = 1 \), and (2.9) can be rewritten as
\[
-2b(b^{p-1} - U_{p-1}(1, b^p) + bU_p(1, b^p)) \equiv (2b - 1)U_{p-1}(1, b^p) \pmod{p^3}.
\]
Thus, we have completed the proof of Theorem 2.

For example, if we take \( b = 2 \) in Theorem 2, we get the congruences in (1.2) and (1.3).

3. References

