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Research Article

Approximate Analytical Solutions of Systems of Fractional Partial Differential Equations

Kesirli Kısmi Diferansiyel Denklem Sistemlerinin Yaklaşık Analitik Çözümleri

Ozan Özkan

Department of Mathematics, Selçuk University, Konya, Turkey

Abstract

In this article, a novel technique which is called Fractional Complex Differential Transformation Method (FCDTM) is proposed for solving systems of fractional partial differential equations (FPDEs). This method is mainly combination of two methods which are Fractional Complex Transform (FCT) and Differential Transform Method (DTM). The efficiency of the new approach is illustrated by applying it successfully to systems of fractional partial differential equations. As a conclusion, the results reveal that the new approach is very effective and simple.

Keywords: Differential transform method, Fractional complex transform, Partial differential equation, Systems of fractional partial differential equations

Öz

Bu makalede, kesirli kısmi diferansiyel denklem sistemlerinin çözümü için Kesirli Kompleks Diferansiyel Dönüşüm Yöntemi olarak adlandırılan yeni bir teknik önerilmektedir. Bu yöntem başlica iki yöntemin, yani Kesirli Kompleks Dönüşüm ve Diferansiyel Dönüşüm Yöntemi yöntemlerinin bir birleşimidir. Bu yeni yaklaşımın etkinliği, kesirli kısmi diferansiyel denklem sistemlerine uygulanarak başarıyla gösterilmiştir. Sonuç olarak, elde edilen sonuçlar bu yeni yaklaşımın çok etkili ve basit olduğunu ortaya koymaktadır.

Anahtar Kelimeler: Diferensiyel dönüşüm yöntemi, Kesirli kompleks dönüşüm, Kısmi türevli diferensiyel denklemler, Kesirli kısmi diferensiyel denklemler

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1. Introduction

Systems of fractional partial differential equations (FPDEs) are mathematical models that are used to describe many phenomena in engineering, physics, chemistry and other sciences. The systems of FPDEs appear in many applications of science and engineering such as fluid dynamics, mathematical biology, chemical kinetics, material science and many other physical processes (Podlubny 1999, Samko et al. 1993). The analytical solution of systems of fractional differential equations has been drawn a lot of attention of specialists and scholars in recent years. As very well-known fact, the exact solution of the most fractional differential equations does not exist, so some numerical techniques have to be used to get approximate solution to this problem. Adomian decomposition method (Hossein and Daftardar-Gejji 2006), variational iteration method (Hossain et al. 2012), homotopy analysis method (Hossein and Seifi 2009), variational homotopy perturbation method (Yanqin 2012) have been found to solve various fractional problems. They are relatively new approaches to provide analytic approximations to linear and nonlinear problems. Among all those method, Differential transform method (DTM) is one of the most used one, due to its simplicity in application, accuracy and efficiency in obtaining exact solutions. DTM is developed by Zhou in 1986 (Zhou 1986). This method transforms the given differential equation into a recurrence equation then constructs the approximate analytical solution in a polynomial form.

*Corresponding Author: oozkan@selcuk.edu.tr

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There are many numerical and analytical methods for solving mathematical problems. Among all these methods, transform is one of the most used one because of the simplicity of its application. Moreover, there are some useful formulas and many transforms to solve that problems, such as Laplace transform (Cassol et al. 2009), the Fourier transform (Sejdic et al. 2011), the integral transform (Cotta and Mikhailov 1993), and the local fractional integral transforms (Yang 2011).

Recently, Fractional Complex Transform (FCT) has become popular. FCT is based on the idea that fractional order differential equations with modified Riemann-Liouville derivatives and Local fractional derivatives can be converted into integer order differential equations, and resultant equations can be solved by advance calculus (He et al. 2012, He 2014, Li and He 2010, Li 2010). Using the same spirit, the aim of this study is to propose a new method by combining FCT with DTM to obtain approximate solutions of systems of FPDEs.

In the current paper the key idea of the new presented method is to convert systems of fractional partial differential equations (FPDEs) into corresponding systems of partial differential equations (PDEs) by using Fractional Complex Transform (FCT). After the transformation, the obtained PDE systems can be solved by DTM which is a very efficient tool for solving such systems.

The paper is structured as follows: in Section 2 the basic idea of Fractional Complex Transformation Method is explained, then in Section 3 the fundamental principle of standard Differential Transform Method is introduced, it will be moved on with presentation of two case studies and a discussion of this results in Section 4, and finally in Section 5 a brief conclusion is given.

2. Fractional Complex Transform

Consider the following general fractional differential equation

$$f(u,u_t^{\alpha},u_x^{\beta},u_t^{2\alpha},u_x^{2\beta},\dots)=0 \tag{1}$$

where $u_t^{\alpha}=\dfrac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}$ describes Local fractional derivative of $u(x,t)$ with $0<\alpha\leq 1$, $0<\beta\leq 1$, (He et al. 2012, He 2014). Now, let introduce the following transforms

$$\begin{cases} T=\dfrac{pt^{\alpha}}{\Gamma(\alpha+1)} \\ X=\dfrac{qx^{\beta}}{\Gamma(\beta+1)} \end{cases} \tag{2}$$

where p and q are unknown constants.

Using that transforms, we can convert fractional derivatives into classical derivatives

$$\begin{cases} \dfrac{\partial^{\alpha}u}{\partial t^{\alpha}}=p\dfrac{\partial u}{\partial T} \\ \dfrac{\partial^{\beta}u}{\partial t^{\beta}}=q\dfrac{\partial u}{\partial X} \end{cases}. \tag{3}$$

Therefore, we can easily convert the fractional partial differential equations into corresponding partial differential equations, so it can be solved with fractional calculus without any difficulty (He et al. 2012, He 2014, Li and He 2010, Li 2010).

3. Differential Transform Method (DTM)

The basic definitions and fundamental operations of the two dimensional differential transform method are defined in Cansu and Özkan 2011, Chen and Ho 1999, Jang et al.2001) as follows:

Definition 1. Let the function $u(x,t)$ be analytic and continuously differentiable function with respect to time t in the domain of interest. Then the two-dimensional differential transform of the function $u(x,t)$ is defined as

$$U(k,h)=\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k\partial t^h}u(x,t)\right]_{\substack{x=0\\t=0}} \tag{4}$$

where $u(x,t)$ is the original function and $U(k,h)$ is the transformed function.

Definition 2. The inverse differential transform of $U(k,h)$ is defined as

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^kt^h \tag{5}$$

combining equations (4) and (5), we obtain

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k\partial t^h}u(x,t)\right]_{\substack{x=0\\t=0}}x^kt^h \tag{6}$$

Equation (6) implies that the concept of the two-dimensional differential transform which is derived from to two-dimensional Taylor series expansion. In this study, we use the lower case letters to represent the original functions and upper case letters to stand for the transformed functions (T-functions). From the above definitions, the fundamental mathematical operations performed by the two-dimensional differential transform can be obtained as (Cansu and Özkan 2011, Chen and Ho 1999, Jang et al.2001) and are listed in Table 1.

Table 1. Some operations of the two dimensional differential transformation.

Original function	Transformed function
$u(x,t)=\alpha(w(x,t)\pm v(x,t))$	$U(k,h)=\alpha(W(k,h)\pm V(k,h))$, α = constant
$u(x,t)=\frac{\partial w(x,t)}{\partial x}$	$U(k,h)=(k+1)W(k+1,h)$
$u(x,t)=\frac{\partial w(x,t)}{\partial t}$	$U(k,h)=(h+1)W(k,h+1)$
$u(x,t)=\frac{\partial^{r+s}w(x,t)}{\partial x^r\partial t^s}$	$U(k,h)=\frac{(k+r)!}{k!}\frac{(h+s)!}{h!}W(k+r,h+s)$
$u(x,t)=w(x,t)v(x,t)$	$U(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}W(r,h-s)V(k-r,s)$
$u(x,t)=x^m t^n$	$U(k,h)=\delta(k-m,h-n)=\begin{cases}1, \text{ for } k=m \text{ and } h=n\\ 0, \text{ otherwise}\end{cases}$

4. Application and Numerical Result

In this section, the applicability of the new approach will be demonstrated by using some examples. All the results are calculated by using Maple.

Example 4.1. Consider the following linear system of FPDEs (Hossein and Seifi 2009): (t>0, x>0, 0<α≤1)

$$\left.\begin{aligned} D_t^\alpha u - v_x + v + u = 0\\ D_t^\alpha v - u_x + v + u = 0\end{aligned}\right\} \tag{7}$$

the initial conditions as:

$$u(x,0)=\sinh(x),\ v(x,0)=\cosh(x) \tag{8}$$

Applying the transformation (He et al. 2012, He 2014) in equation (7); for simplicity we set $p=1$ so we get the following systems of PDE.

$$\left.\begin{aligned} u_T - v_x + v + u = 0\\ v_T - u_x + v + u = 0\end{aligned}\right\} \tag{9}$$

which can be solved via DTM. Applying the DTM to both sides of the Equations (8) and (9), we obtain the following recursive formula

$$\left.\begin{aligned}(h+1)U(k,h+1)-(k+1)V(k+1,h)+V(k,h)+U(k,h)=0\\ (h+1)V(k,h+1)-(k+1)U(k+1,h)+V(k,h)+U(k,h)=0\end{aligned}\right\} \tag{10}$$

using the initial condition (8), we get

$$\left.\begin{aligned} U(k,0)=\begin{cases}0, & k=0,2,4,...\\ \frac{1}{k!}, & k=1,3,5,...\end{cases}\\ V(k,0)=\begin{cases}0, & k=1,3,5,...\\ \frac{1}{k!}, & k=0,2,4,...\end{cases}\end{aligned}\right\} \tag{11}$$

Components of the spectrums of u and v, $U(k,h)$ and $V(k,h)$, can be calculated by utilizing the recurrence relation (10) and the transformed initial conditions (11). From the inverse transform of DTM given by equation (5) and via equation (6), we get the approximate solution of the systems (8)-(9) in series form

$$u(x,T)=\left(\sum_{k=1,3,...}^{\infty}\frac{x^k}{k!}\right)\left(1+\frac{T^2}{2!}+\frac{T^4}{4!}+...\right)-\left(\sum_{k=0,2,...}^{\infty}\frac{x^k}{k!}\right)\left(T+\frac{T^3}{3!}+\frac{T^5}{5!}+...\right) \tag{12}$$

$$v(x,T)=\left(\sum_{k=0,2,...}^{\infty}\frac{x^k}{k!}\right)\left(1+\frac{T^2}{2!}+\frac{T^4}{4!}+...\right)-\left(\sum_{k=1,3,...}^{\infty}\frac{x^k}{k!}\right)\left(T+\frac{T^3}{3!}+\frac{T^5}{5!}+...\right) \tag{13}$$

The inverse transformation of FCT will yield the solution in series form which is given by,

$$u(x,t)=\left(\sum_{k=1,3,...}^{\infty}\frac{x^k}{k!}\right)(1+\frac{t^{2\alpha}}{2!\Gamma^2(\alpha+1)}+\frac{t^{4\alpha}}{4!\Gamma^4(\alpha+1)}+...)-\left(\sum_{k=0,2,...}^{\infty}\frac{x^k}{k!}\right)(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{3\alpha}}{3!\Gamma^3(\alpha+1)}+\frac{t^{5\alpha}}{5!\Gamma^5(\alpha+1)}+...) \tag{14}$$

$$v(x,t)=\left(\sum_{k=0,2,...}^{\infty}\frac{x^k}{k!}\right)(1+\frac{t^{2\alpha}}{2!\Gamma^2(\alpha+1)}+\frac{t^{4\alpha}}{4!\Gamma^4(\alpha+1)}+...)-\left(\sum_{k=1,3,...}^{\infty}\frac{x^k}{k!}\right)(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{3\alpha}}{3!\Gamma^3(\alpha+1)}+\frac{t^{5\alpha}}{5!\Gamma^5(\alpha+1)}+...) \tag{15}$$

Setting $\alpha=1$, problem (7)-(8) reduces to system of linear PDE (Wazwaz 2007):

$$\left.\begin{array}{l} u_t - v_x + v + u = 0 \\ v_t - u_x + v + u = 0 \end{array}\right\} \quad (16)$$

with the initial condition

$u(x,0) = \sinh(x)$, $v(x,0) = \cosh(x)$. (17)

We ultimately obtain the closed form solutions as below. These are exact solutions confirmed by (Wazwaz 2007).

$u(x,t) = \sinh(x-t)$, $v(x,t) = \cosh(x-t)$. (18)

Example 4.2. Consider the following time-fractional coupled Burgers equations (Yanqin 2012):

$$\left.\begin{array}{l} D_t^\alpha u - u_{xx} - 2uu_x + (uv)_x = 0 \\ D_t^\alpha v - v_{xx} - 2vv_x + (uv)_x = 0 \end{array}\right\} \quad (19)$$

where ($t>0$, $x>0$, $0<\alpha\leq 1$) subject to the initial conditions:

$u(x,0) = \sin(x)$, $v(x,0) = \sin(x)$. (20)

Using the method which is employed in Example 4.1, we obtain the following solutions:

$$\begin{array}{l} u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k t^h = (\frac{x}{1!} - \frac{x^3}{3!} + \ldots) \\ (1 - \frac{t^\alpha}{1!\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{2!\Gamma^2(\alpha+1)} - \ldots) \end{array} \quad (21)$$

$$\begin{array}{l} v(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} V(k,h)x^k t^h = (\frac{x}{1!} - \frac{x^3}{3!} + \ldots) \\ (1 - \frac{t^\alpha}{1!\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{2!\Gamma^2(\alpha+1)} - \ldots) \end{array} \quad (22)$$

If $\alpha = 1/2$, then we get

$u(x,t) = \sin(x)E_{1/2}(-t^{1/2})$, $v(x,t) = \sin(x)E_{1/2}(-t^{1/2})$ (23)

where $E_{1/2}(-t^{1/2})$ is the Mittag-Leffler function (Mittag-Leffler 1903). As a special case when $\alpha = 1$, problem (19)-(20) reduces coupled Burgers equations (Yanqin 2012):

$$\left.\begin{array}{l} u_t - u_{xx} - 2uu_x + (uv)_x = 0 \\ v_t - v_{xx} - 2vv_x + (uv)_x = 0 \end{array}\right\} \quad (24)$$

and the closed form solution is

$u(x,t) = \sin(x)e^{-t}$, $v(x,t) = \sin(x)e^{-t}$. (25)

The convergence and the accuracy of the solution series in (21)-(22) when it is truncated at level $n \in \mathbb{N}$ are analyzed by calculation the absolute errors E_n which is defined as follows:

$$E_n = |u_{exact}(x,t) - u_n(x,t)| \quad (26)$$

where $u_i(x,t) = \sum_{k=0}^{i}\sum_{h=0}^{i} U(k,h)x^k t^h, i \in \mathbb{N}$ and $u_{exact}(x,t)$ denotes the exact solution of the equation.

In Figures 1(a)-1(b), we have presented exact solution and approximate solution at $\alpha = 1$, i = 12. It can be seen from Figures 1(a)-1(b) that the solutions obtained by the new approach are nearly identical with the exact solutions. Also; Figures 2 shows the error E_{12} of approximate solutions at $\alpha = 1$. We can say that E_n is decreasing by n. So convergence of the method is promising in numerically. For theoretical convergence further analysis is required.

6. Conclusion

In this paper, we suggested an effective modification of the differential transform method called the Fractional Complex Differential Transformation Method (FCDTM) that combines Fractional Complex Transform (FCT)

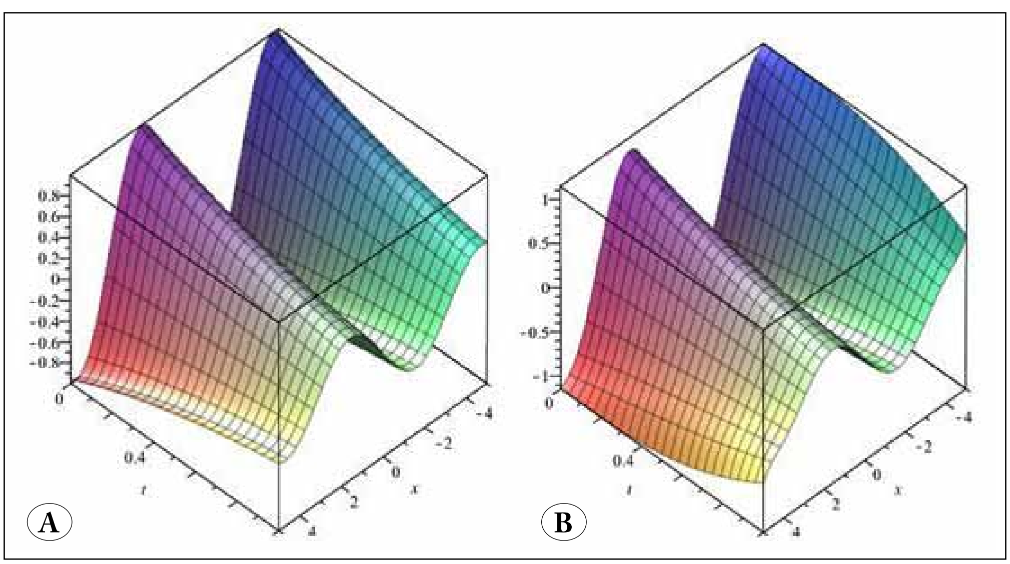

Figure 1. The surface shows the solutions for (19)-(20):
(A) exact solution *u(x,t)* or *v(x,t)*;
(B) numerical solution *i*=12, when $\alpha = 1$.

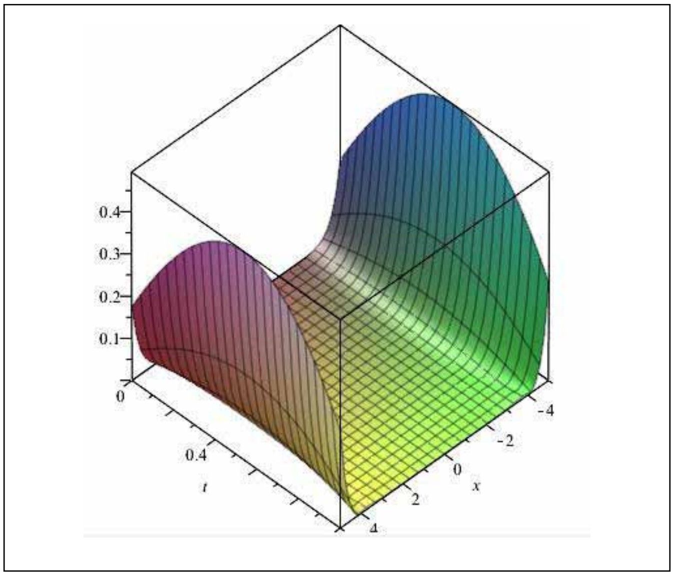

Figure 2. $E_{12} = |u_{exact}(x,t) - u_{12}(x,t)|$ or $E_{12} = |v_{exact}(x,t) - v_{12}(x,t)|$, $\alpha = 1$.

and Differential Transform Method (DTM) to obtain approximate solution of systems of fractional partial differential equation. Systems of fractional partial differential equation can be easily converted systems of partial differential equation by fractional complex transform, so that it's differential transform algorithm can be simply constructed. The fractional complex differential transformation is extremely simple and yet effective for solving fractional differential equations. The method is accessible to all with basic knowledge of Advance Calculus and with little Fractional Calculus. The results confirm that FCDTM is very powerful and efficient technique in finding approximate solutions for systems of fractional partial differential equations.

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