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Research Article

The Neighbor Toughness of Wheels and Related Networks

Tekerlek ve Benzer Grafların Komşu Dayanıklılık Değerleri

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Abstract

In a network (communication, computer, transportation), the vulnerability parameters measure the resistance of the network to disruption of operation after the failure of certain stations or communication links. The neighbor toughness of a graph G is defined as $NT(G) = min\left\{\frac{|S|}{\omega(G/S)} : \omega(G/S) \geq 1\right\}$, where S is any vertex subversion strategy of G and ω(G/S) is the number of connected components in the graph G/S. In this paper, the relations between neighbor toughness and other parameters are determined. Next, the neighbor toughness of wheels and some related networks namely gear, helm, sunflower and friendshipgraph are calculated.

Keywords: Graph theory, Neighbor toughness, Toughness, Vulnerability

Öz

Bir (iletişim, bilgisayar, taşıma) ağda, bazı merkezlerin veya bağlantı hatlarının bozulmasıyla iletişim kesilene kadar ağın gösterdiği dayanma gücünün ölçümüne ağın zedelenebilirlik değeri denir. Bir G grafi için S, komşu kesim küme ve ω(G/S) S kümesi ve komşu tepeleri graftan atıldıktan sonra bileşen sayısı olmak üzere; komşu dayanıklılık değerinin tanımı $NT(G) = min\left\{\frac{|S|}{\omega(G/S)} : \omega(G/S) \geq 1\right\}$, şeklindedir. Bu çalışmada komşu dayanıklılık sayısı ile diğer graf parametreler arasında bağlantı incelenmiştir. Ardından, tekerlek ve benzer grafların komşu dayanıklılık değerleri hesaplanmıştır.

Anahtar Kelimeler: Çizge teori, Komşu dayanıklılık değeri, Dayanıklılık değeri, Zedelenebilirlik

MSC2010: 05C40, 68M10, 68R10

1. Introduction

Let $G = (V,E)$ be a graph and u a vertex in G. We call $N(u) = \{v : v \in V(G), uv \in E(G)\}$ the open neighborhood of u, and $N[u] = \{u\} \cup N(u)$ its closed neighborhood. We define analogously the open neighborhood $N(S)$ = for any $S \subseteq V(G)$ and the closed eighborhood $N[S]$ = for any $S \subseteq V(G)$. A vertex $u \in V(G)$ is subverted when the closed neighborhood $N[u]$ is deleted from G. A vertex subversion strategy $S \subseteq V(G)$ is a set of vertices whose closed neighborhood is deleted from G. The survival subgraph G/S is the subgraph obtained by the subversion strategy S applied to G, *i.e.*, $G/S = G - N[S]$. S is called a cut-strategy of G if the survival subgraph G/S is disconnected, clique or an empty set. Let $deg(u)$ denote the degree of the vertex u in G.

It is known that communication systems are often exposed to failures and attacks. So robustness of the network topology is a key aspect in the design of networks.

The stability of a communication network, composed of processing nodes and communication links, is of prime importance to network designers. As the network begins losing links or nodes, eventually there is a loss in its effectiveness. In the literature, various measures were defined to measure the robustness of network and a variety of graph theoretic parameters have been used to derive formulas to calculate network vulnerability. Parameters used to measure the vulnerability include connectivity, integrity

(Bagga et al, 1992), scattering number (Jung 1978), and toughness (Chvatal 1973). Recent interest in the

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vulnerability and reliability of networks (communication, computer, transportation) has given rise to a host of other measures, some of which are more global in nature; see, for example, (Kırlangıç and Aytac 2004, Turacı 2016, Turacı and Ökten 2015, Aslan and Bacak-Turan 2016).

However, most of these parameters do not consider the neighborhoods of the affected vertices. On the other hand, in spy networks, if a spy or a station is captured, then adjacent stations are unreliable. Therefore, neighborhoods should be taken into consideration in spy networks. Nevertheless, there are very few parameters concerning neighborhoods such as neighbor connectivity (Gunther 1985), neighbor integrity (Cozzens and Wu 1996), and neighbor scattering number (Wei et al. 2011), neighbor rupture degree (Bacak-Turan and Kirlangic 2013) and neighbor toughness (Kurkcu and Aksan 2016).

The most common vulnerability parameters concerning to spy networks are as follows.

The neighbor connectivity of a graph G is

$$\varkappa(G) = min\{|S|\}, \tag{1}$$

where S is a cut-strategy of G (Gunther 1985).

The neighbor integrity of a graph G is defined to be

$$NI(G) = min\{|S| + c(G/S)\}, \tag{2}$$

where S is any cut- strategy of G and $c(G/S)$ is the order of the largest connected component of G/S (Cozzens and Wu 1996).

The vertex neighbor scattering number of a graph G is defined as

$$S(G) = max\left\{\begin{array}{l}\omega(G/S) - |S| : S \text{ is a cut} - \text{strategy of } G, \\ \omega(G/S) \geq 1\end{array}\right\} \tag{3}$$

where $\omega(G/S)$ denotes the number of connected components in G/S (Wei et al. 2011). The neighbor rupture degree of a noncomplete graph G is defined to be

$$Nr(G) = max\left\{\begin{array}{l}\omega(G/S) - |S| - c(G/S) : S \subset G, \\ \omega(G/S) \geq 1\end{array}\right\} \tag{4}$$

where S is any vertex subversion strategy of G, $\omega(G/S)$ is the number of connected

components in G/S and $c(G/S)$ is the maximum order of the components of G/S (Bacak-Turan and Kirlangic 2013).

The neighbor toughness of a graph G is defined as

$$NT(G) = min\left\{\frac{|S|}{\omega(G/S)} : \omega(G/S) \geq 1\right\} \tag{5}$$

where S is a cut- strategy of G and $w(G/S)$ is the number of connected components in the graph G/S (Kurkcu and Aksan 2016). In particular, the neighbor toughness of a complete graph K_n is defined to be 0. A cut-strategy S of G is called an *NT-set* of G if

$$NT(G) = \frac{|S|}{\omega(G/S)}.$$

Now, we give some lower and upper bounds for the neighbor toughness in terms of well known graph parameters.

Theorem 1. (Kurkcu and Aksan 2016) *Let G be a connected graph of order n. Then,*

$$NT(G) \leq K(G). \tag{6}$$

Theorem 2. (Kurkcu and Aksan 2016) *Let G be a connected graph of order n. Then,*

$$NT(G) \leq \delta(G). \tag{7}$$

Theorem 3. (Kurkcu and Aksan 2016) *For any graph G,*

$$NT(G) \leq NI(G). \tag{8}$$

Next, the neighbor toughness of some graphs are listed.

Theorem 4. (Kurkcu and Aksan 2016) *Let P_n be a path graph of order n(≥3). Then,*

$$NT(P_n) = \begin{cases} 1, \text{if } n = 3,4; \\ \frac{1}{2}, \text{if } n \geq 5. \end{cases} \tag{9}$$

Theorem 5. (Kurkcu and Aksan 2016) *Let C_n be a cycle graph of order n(≥4). Then,*

$$NT(C_n) = \begin{cases} 2, \text{if } n = 6,7; \\ 1, \text{if } n = 4,5 \text{ or } n \geq 8. \end{cases} \tag{10}$$

Theorem 6. (Kurkcu and Aksan 2016) *Let $K_{m,n}$ be a bipartite graph. Then,*

$$NT(K_{m,n}) = \begin{cases} \frac{1}{m-1}, \text{if } n < m; \\ \frac{1}{n-1}, \text{if } n \geq m. \end{cases} \tag{11}$$

Corollary 1. (Kurkcu and Aksan 2016) *The neighbor toughness of the star graph $K_{1,n}$ is*

$$NT(K_{1,n}) = \frac{1}{n-1}. \tag{12}$$

In Section 2, we give theorems related to neighbor toughness and graph parameters. In Section 3, the neighbor toughness of wheel graphs, gear graphs, helm graphs, sunflower graphs and friendship graphs are calculated.

2. Bounds

In this section, we will give lower bound about the neighbor toughness.

Theorem 7. *Let G be a connected graph of order n. Then,*

$$NT(G) \geq \frac{1}{n-2}. \tag{13}$$

Proof: Let S be a cut-strategy of G. It is clear that for any S of G, we have $|S| \geq 1$, and $w(G/S) \leq n$-2.

Therefore,

$$\frac{|S|}{\omega(G/S)} \geq \frac{1}{n-2}. \tag{14}$$

By the definition of neighbor toughness, we have

$$NT(G) \geq \frac{1}{n-2}. \tag{15}$$

Theorem 8. *If G is a noncomplete graph of order n, independence number* $\beta(G)$ *and neighbor connectvity* $\kappa(G)$*, then*

$$NT(G) \geq \frac{K(G)}{\beta(G)}. \tag{16}$$

Proof: Let S be a cut-strategy of G. We have $|S| \geq K(G)$ and $w(G/S) \leq B(G)$ for any graph G. So

$$\frac{|S|}{\omega(G/S)} \geq \frac{K(G)}{\beta(G)}. \tag{17}$$

From the definition of neighbor toughness, we have

$$NT(G) \geq \frac{K(G)}{\beta(G)}. \tag{18}$$

Theorem 9. *Let G be a connected graph of order n and neighbor connectvity* $\kappa(G)$*, then*

$$NT(G) > \frac{K(G)}{n-K(G)}. \tag{19}$$

Proof: Let S be a cut-strategy of G. For any S of G, we have $|S| \geq \kappa(G)$, and $w(G/S) \leq n-\kappa(G)$.

Therefore,

$$\frac{|S|}{\omega(G/S)} \geq \frac{K(G)}{n-K(G)} \tag{20}$$

$$NT(G) > \frac{K(G)}{n-K(G)}. \tag{21}$$

3. The Neighbor Toughness of Some Graphs

In this section, we calculate the neighbor toughness of wheel graphs, gear graphs, helm graphs, friendship graphs, and sunflower graphs.

Definition 1. (Harary 1994) The wheel graph with n spokes, W_n, is the graph that consists of an n-cycle and one additional vertex, say u, that is adjacent to all the vertices of the cycle. In Figure 1 we display W_6.

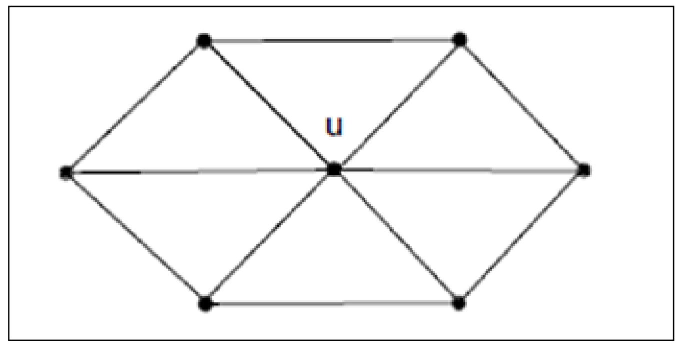

Figure 1. W_6 wheel graph.

Theorem 10. *Let* W_n *be a wheel graph of order n(≥4). Then,*

$$NT(W_n) = 1. \tag{22}$$

Proof: We assume $n \geq 4$. A wheel graph W_n is the join of a cycle and the complete graph K_1, $W_n = K_1 + C_{n-1}$. Let S be a cut-strategy of W_n. Then, the set of S contains only some of the vertices belonging to the cycle, otherwise it contradicts to $w(W_n/S) \geq 1$. If $v \in V(C_{n-1})$ and $S=\{v\}$, then we get $W_n/S = P_{n-4}$. If $S \subset V(C_{n-1})$ and $|S|=r+1$, then we have

$w(W_n/S) \leq r+1$. Hence,

$$\frac{|S|}{\omega(W_n/S)} \geq \frac{r+1}{r+1} = 1. \tag{23}$$

So we have, $NT(W_n) \geq 1$ (24)

It is obvious that there is a cut strategy of W_n such that $|S|=2$ then we have $w(W_n/S)=2$. From the definition of neighbor toughness we have,

$$NT(W_n) = 1. \tag{25}$$

Thus by (23) and (24),

$$NT(W_n) = 1. \tag{26}$$

Definition 2. (Gallian 2007) The gear graph is a wheel graph with a vertex added between each pair adjacent graph vertices of the outer cycle. The gear graph G_n has $2n$+1 vertices and $3n$ edges. In Figure 2 we display G_6.

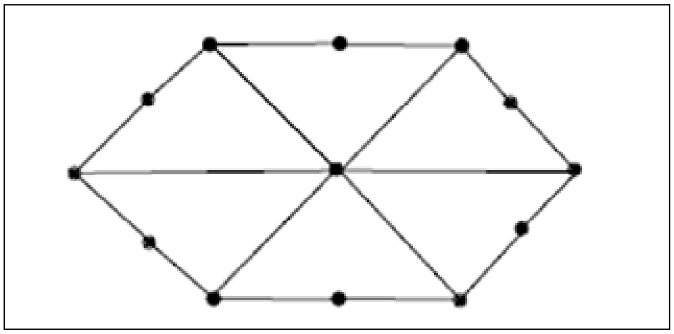

Figure 2. G_6 gear graph.

Theorem 11. *Let G_n be a gear graph of order n(≥3). Then,*

$$NT(G_n) = \frac{1}{n}. \tag{27}$$

Proof: Let S be a cut-strategy of G_n, $|S| = r$. Let $u \in V(G_n)$ and $deg(u) = n$. If $r \geq 1$, then we have $w(G_n/S) \leq n$. So,

$$\frac{|S|}{w(G_n/S)} \geq \frac{r}{n}. \tag{28}$$

The function $f(r) = \frac{r}{n}$ is an increasing function and it takes its minimum value at $r = 1$ and we have

$$NT(G_n) \geq \frac{1}{n}. \tag{29}$$

It is obvious that there is a cut strategy S of G_n such that $S=\{u\}$, then we have

$w(G_n/S) = n$. Hence $\quad NT(G_n) = \frac{1}{n}.$ (30)

Definition 3. (Javaid and Shokat 2008**)** The helm graph H_n is the graph obtained from an n-wheel graph by adjoining a pendant edge at each node of the cycle. The helm graph H_n has $2n+1$ vertices and $3n$ edges. In Figure 3 we display H_4.

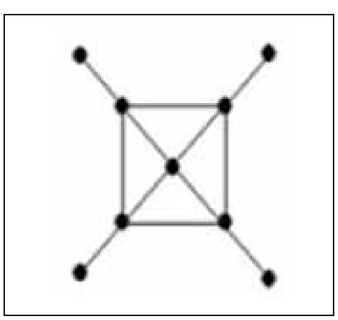

Figure 3. H_4 helm graph.

Theorem 12. *Let H_n be a helm graph of order n(≥4). Then,*

$$NT(H_n) = \frac{1}{n}. \tag{31}$$

Proof: The proof is similar to Theorem 11.

Definition 4. (Javaid and Shokat 2008**)** The sunflower graph SF_n is defined as follows: consider a wheel with central vertex c and an n-cycle $v_0, v_1, v_2, \ldots, v_{n-1}$ and additional n vertices $w_0, w_1, w_2, \ldots, w_{n-1}$ where w_i is joined by edges to v_i, v_{i+1} for $i = 0, 1, 2, \ldots, n-1$ where $i+1$ is taken modulo n. SF_n has order $2n + 1$ and size $4n$. In Figure 4 we display SF_4.

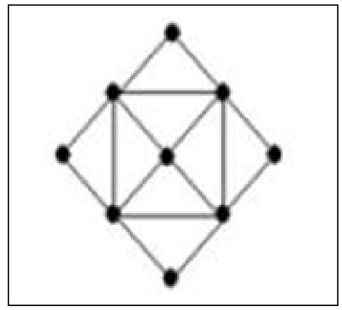

Figure 4. SF_4 sunflower graph.

Theorem 13. *Let SF_n be a sunflower graph of order n(≥4). Then,*

$$NT(SF_n) = \frac{1}{n}. \tag{32}$$

Proof: The proof is similar to Theorem 11.

Definition 5. (Gallian 2007) The friendship graph F_n is a planar graph with *2n+1* vertices and *3n* edges. In Figure 5 we display F_4.

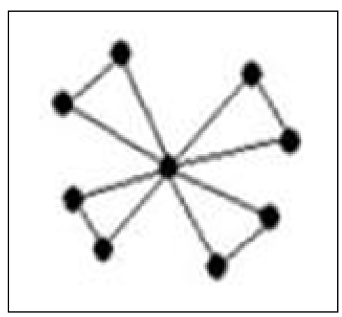

Figure 5. F_4 friendship graph.

Theorem 14. *Let SF_n be a sunflower graph of order n(≥4). Then,*

$$NT(F_n) = \frac{1}{n-1}. \tag{33}$$

Proof: Let S be a cut-strategy of F_n and $|S| = r$ be the number of removing vertices from F_n. Let $u \in V(F_n)$ and $deg(u) = 2$. If $r \geq 1$, then we have $w(F_n/S) \leq n-1$. So,

$$\frac{|S|}{\omega(F_n/S)} \geq \frac{r}{n-1}. \tag{34}$$

The function $f(r) = \frac{r}{n-1}$ is an increasing function and it takes its minimum value at $r = 1$ and we have

$$NT(Fn) \geq \frac{1}{n-1}. \tag{35}$$

It is obvious that there is a cut strategy S of G_n such that

$S=\{u\}$, then we have

$w(F_n/S) = n-1$. Hence $NT(F_n) = \frac{1}{n-1}$. (36)

4. Conclusion

Reliability and efficiency are important criteria in the network design. When we design a network, we want it to be as stable as possible. In this study, it was shown that the neighbor toughness of whell graphs is always a constant value. On the other hand, it was proven that the neighbor toughness of gear graphs, helm graphs, friendship graphs and sunflower graphs are not constant, they depend on the vertex number.

$NT(W_n) = 1, but\ NT(G_n) = NT(H_n) = NT(SF_n) = \frac{1}{n}\ and$

$NT(F_n) = \frac{1}{n-1}$.

Therefore if we want to choose the most stable graph among these graphs, we need to choose wheel graphs. That is to choose the graph with maximum neighbor toughness.

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