Inequalities via \( n \)-Times Differentiable Quasi-Convex Functions

\( n \)-Mertebeden Türevlenebilir Quasi-Konveks Fonksiyonlar Yardımıyla Eşitsizlikler

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Abstract

In this paper, some integral inequalities for \( n \)-times differentiable quasi-convex functions are established.

Keywords: Hermite–Hadamard inequality, Hölder inequality, Quasi-convex functions, Power-mean inequality

1. Introduction

A function \( f:[a,b] \rightarrow \mathbb{R} \) is said to be quasi-convex function on \([a,b]\) if the inequality \( f(ax + (1-a)y) \leq \max\{f(x), f(y)\} \) holds for all \( x, y \in [a,b] \) and \( \alpha \in [0,1] \). For some results about quasi-convexity see [1], [5], [7], [8], [13].

Hermite–Hadamard inequality is defined as below:

Let \( f:I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a convex function on an interval \( I \) and \( a, b \in I \) with \( a < b \). Then

\[
\int_a^b f\left(\frac{a+b}{2}\right) \leq \frac{b-a}{2} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}
\]

holds. Both inequalities hold in the reversed direction if \( f \) is concave. For some generalizations and applications concerning Hermite–Hadamard inequality see [1], [3], [6], [7], [9]-[12], [16] and [17].

Recently, in the literature there are so many manuscripts about \( n \)-times differentiable functions on several kinds of convexities. In references [3], [4], [6], [9], [10] and [14]-[16] readers can find some results about this issue.

The main aim of this paper is to prove new integral inequalities for \( n \)-times differentiable quasi-convex functions.

To prove our main results, we use the following lemmas.

Lemma 1.1. [2] Let \( f:[a,b] \rightarrow \mathbb{R} \) be a mapping such that the derivative \( f^{(n+1)}(n \geq 1) \) is absolutely continuous on \([a,b]\). Then for any \( x \in [a,b] \) one has the equality:

\[
\int_a^b f(t) \, dt = \sum_{k=0}^{\infty} \frac{1}{k+1} \left[ (x-a)^{k+1} f^{(k)}(a) + (b-x)^{k+1} f^{(k)}(b) \right] + \int_a^b (x-t)^{k} f^{(k)}(t) \, dt.
\]

Lemma 1.2. [6] Let \( f: \mathbb{R} \rightarrow \mathbb{R} \) be a mapping such that the derivative \( f^{n+1}(n \geq 1) \) exists on \( \mathbb{R} \) and \( f^{n+1} \in L(a,b) \) for \( n \geq 1 \). We have the following identity:

\[
\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx = \frac{1}{(k+1)!} \int_0^1 t^{k} (n-t)^{k} f^{(n)}(t) \, dt.
\]

2. Inequalities for \( n \)-Times Differentiable Quasi-Convex Functions Via Lemma 1.1.

The first result is given in the following theorem:

Theorem 2.1. Let \( f:[a,b] \rightarrow \mathbb{R} \) be a mapping such that the derivative \( f^{(n+1)}(n \geq 1) \) is absolutely continuous on \([a,b]\). If \( f^{(n)} \) is quasi-convex on \([a,b]\), following inequality holds

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\[
\int_{t_0}^{t_1} f(t) \, dt - \frac{1}{(k+1)!} \left( \frac{(-1)^{i+1} f^{(i)}(b)}{(b-x)^{i+1}} \right) \left( b-a \right)^{i+1} \frac{1}{n+1} + 1
\]
for all \( t \in [a,b] \).

**Proof.** From Lemma 1.1., property of the modulus, well-known Hölder integral inequality and quasi-convexity of \( f^m \), we can write
\[
\int_{t_0}^{t_1} f(t) \, dt - \frac{1}{(k+1)!} \left( \frac{(-1)^{i+1} f^{(i)}(b)}{(b-x)^{i+1}} \right) \left( b-a \right)^{i+1} \frac{1}{n+1} + 1
\]
which completes the proof.

3. **Inequalities for **n**-Times Differentiable Quasi-Convex Functions Via Lemma 1.2.**

**Theorem 3.1.** Let \( f_t : [a,b] \to \mathbb{R} \) be an \( n \)-times differentiable function. If \( f^m \) is quasi-convex on \( [a,b] \), following inequality holds
\[
\left| f(a) + f(b) \right| \leq \frac{1}{2} \int_{a}^{b} f(x) \, dx - \sum_{i=2}^{n} \frac{(k-1)(b-a)^{k}}{2(k+1)!} f^{(i)}(a)
\]
for \( q \geq 1 \) and \( n \geq 2 \).

**Proof.** From Lemma 1.2., property of the modulus, well-known power-mean integral inequality and quasi-convexity of \( f^m \), we can write
\[
\int_{t_0}^{t_1} f(t) \, dt - \frac{1}{(k+1)!} \left( \frac{(-1)^{i+1} f^{(i)}(b)}{(b-x)^{i+1}} \right) \left( b-a \right)^{i+1} \frac{1}{n+1} + 1
\]
which completes the proof.

**Theorem 3.2.** Let \( f_t : [a,b] \to \mathbb{R} \) be an \( n \)-times differentiable function. If \( f^m \) is quasi-convex on \( [a,b] \), following inequality holds
\[
\left| f(a) + f(b) \right| \leq \frac{1}{2} \int_{a}^{b} f(x) \, dx - \sum_{i=2}^{n} \frac{(k-1)(b-a)^{k}}{2(k+1)!} f^{(i)}(a)
\]
for \( q \geq 1 \) and \( n \geq 2 \).
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx - \sum_{i=1}^{n-1} \left( \frac{(k-1)(b-a)^k}{2(k+1)!} \right) f^{(i)}(a) \right|
\]
\[
\leq \frac{(b-a)^n}{2n!} \left( \frac{1}{np - p + 1} \right)^{n+1} \left( \frac{n^{p+1} - (n-2)^{p+1}}{q+1} \right)^{\frac{1}{p+1}}
\]
\[
\times \max \left\{ \left| f^{(i)}(a) \right|, \left| f^{(i)}(b) \right| \right\}^{\frac{1}{p+1}}
\]
for \( n \geq 2, q > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Proof:** From Lemma 1.2., property of the modulus, well-known Hölder integral inequality and quasi-convexity of \( |f^{(i)}| \) we can write

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx - \sum_{i=1}^{n-1} \left( \frac{(k-1)(b-a)^k}{2(k+1)!} \right) f^{(i)}(a) \right|
\]
\[
\leq \frac{(b-a)^n}{2n!} \int_0^1 t^{n-1} (n-2t) \left| f^{(i)}(ta + (1-t)b) \right| \, dt
\]
\[
\leq \frac{(b-a)^n}{2n!} \left( \frac{1}{np - p + 1} \right)^{n+1} \left( \frac{n^{p+1} - (n-2)^{p+1}}{q+1} \right)^{\frac{1}{p+1}}
\]
\[
\times \max \left\{ \left| f^{(i)}(a) \right|, \left| f^{(i)}(b) \right| \right\}^{\frac{1}{p+1}}
\]

The proof is completed.

### 4. References


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