On the Quadra Lucas-Jacobsthal Numbers

Quadra Lucas-Jacobsthal Sayıları Üzerine

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Abstract

In this paper, we define the Quadra Lucas-Jacobsthal numbers and then, we give some properties of this sequences. Moreover, we obtain spectral norms of circulant matrices with Quadra Lucas-Jacobsthal numbers.

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1. Introduction

Let \( p \) and \( q \) be non-zero integers. The second order linear recurrences of the Fibonacci and Lucas types are defined as follows:

\[
U_n = pU_{n-1} + qU_{n-2}, \quad V_n = pV_{n-1} + qV_{n-2}
\]

where \( U_0 = 0, U_1 = 1 \) and \( V_0 = 2, V_1 = p \).

Fibonacci, Lucas, Pell, Pell–Lucas, Jacobsthal and Jacobsthal-Lucas numbers can be derived from (1) and (2). For \( p = q = 1 \), \( U_n = F_n \) where \( F_n \) is the \( n \)-th Fibonacci number. For \( p = 2, \ q = 1 \), \( U_n = P_n \) where \( P_n \) is the \( n \)-th Pell number. Similarly, \( p = 1 \) and \( q = 2 \), \( U_n = J_n \) where \( J_n \) is the \( n \)-th Jacobsthal number.

On the other hand, for \( p = q = 1 \), \( V_n = L_n \) where \( L_n \) is the \( n \)-th Lucas number. For \( p = 2 \) and \( q = 1 \), \( V_n = Q_n \) where \( Q_n \) is the \( n \)-th Pell-Lucas number. Similarly, for \( p = 1 \) and \( q = 2 \), \( V_n = J_n \) where \( J_n \) is the \( n \)-th Jacobsthal-Lucas number.

The characteristic equation of \( U_n \) and \( V_n \) is \( x^2 - px - q = 0 \) and the roots are

\[
x_1 = \frac{p + \sqrt{p^2 + 4q}}{2}, \quad x_2 = \frac{p - \sqrt{p^2 + 4q}}{2}.
\]

The generating functions of \( U_n \) and \( V_n \) are

\[
U(x) = \frac{x}{1-px-qx^2}, \quad V(x) = \frac{2-px}{1-px-qx^2}.
\]

Some authors have studied certain integer sequences. For example in [5], Taşcı defined Quadrapell numbers by the following recurrence relation for \( n \geq 4 \),

\[
D_n = D_{n-2} + 2D_{n-3} + D_{n-4}
\]

with the initial values \( D_0 = D_1 = D_2 = 1 \) and \( D_3 = 2 \). Then the author gave some algebraic identities for the Quadrapell numbers. In [4], Özkoç defined a Quadra Fibona-Pell numbers by the following recurrence relation for \( n \geq 4 \),

\[
W_n = 3W_{n-1} - 3W_{n-3} - W_{n-4}
\]

with the initial values \( W_0 = W_1 = 0, W_2 = 1, W_3 = 3 \). Then the author gave some properties of Quadra Fibona-Pell sequences. In [6], the authors studied the integer sequence

\[
T_n = -5T_{n-1} - 5T_{n-2} + 2T_{n-3} + 2T_{n-4}
\]

with the initial values \( T_0 = T_1 = 0, T_2 = -3 \) and \( T_3 = 12 \).
In this paper, we define a similar sequence related to Lucas and Jacobsthal numbers. Then we give some properties for these numbers.

2. Quadra Lucas-Jacobsthal Numbers

**Definition 2.1.** The Quadra Lucas-Jacobsthal numbers $S_n$ are defined by the following recurrence relation for $n \geq 4$,

$$S_n = 2S_{n-1} + 2S_{n-2} - 3S_{n-3} - 2S_{n-4}$$

(4)

with the initial values $S_0 = 1$, $S_1 = 2$, $S_2 = 4$ and $S_3 = 7$.

We note that the characteristic equation (4) is $x^4 - 2x^3 - 2x^2 + 3x + 2 = 0$ and thus the roots of it are

$$\alpha = \frac{1 + \sqrt{5}}{2}, \beta = \frac{1 - \sqrt{5}}{2}, \gamma = 2 \text{ and } \delta = -1$$

where $\alpha, \beta$ are the roots of the characteristic equation of Lucas numbers and $\gamma, \delta$ are the roots of the characteristic equation of Jacobsthal numbers. Now we give the generating function of $S_n$.

**Theorem 2.1.** The generating function for $S_n$ is

$$S(x) = \frac{x^3 - 4x^2 - 2x + 2}{2x^4 + 3x^3 - 2x^2 - 2x + 1}.$$ 

**Proof.** The generating function $S(x)$ has the form

$$S(x) = \sum_{n=0}^{\infty} S_n x^n = S_0 + S_1 x + S_2 x^2 + \cdots$$

Since the characteristic equation of (4) is $x^4 - 2x^3 - 2x^2 + 3x + 2 = 0$, we have

$$(2x^4 + 3x^3 - 2x^2 - 2x + 1)S(x) = (2x^4 + 3x^3 - 2x^2 - 2x + 1)(S_0 + S_1 x + S_2 x^2 + \cdots + S_n x^n)$$

$$(2x^4 + 3x^3 - 2x^2 - 2x + 1)S(x) = S_0 + (S_1 - 2S_0)x + \cdots + (S_n - 2S_{n-1} - 2S_{n-2} + 3S_{n-3} + 2S_{n-4})x^n + \cdots$$

From the (4), we get

$$(2x^4 + 3x^3 - 2x^2 - 2x + 1)S(x) = x^3 - 4x^2 - 2x + 2.$$ 

So

$$S(x) = \frac{x^3 - 4x^2 - 2x + 2}{2x^4 + 3x^3 - 2x^2 - 2x + 1}.$$ 

which is desired result.

**Theorem 2.2.** Let $S_n$ be the Quadra Lucas-Jacobsthal numbers. For $n \geq 0$, the Binet formula for $S_n$ is

$$S_n = (\alpha^n + \beta^n) + \left(\frac{\gamma^n - \delta^n}{\gamma - \delta}\right).$$

**Proof.** We can write $S(x)$ as

$$S(x) = \frac{2 - x}{1 - x - x^2} + \frac{x}{1 - x - 2x^2}.$$ 

(5)

From (3), the generating function of Lucas and Jacobsthal numbers are

$$L(x) = \frac{2 - x}{1 - x - x^2}, J(x) = \frac{x}{1 - x - 2x^2}$$

respectively. From (5) and (6), we have

$$S(x) = L(x) + J(x).$$

Hence we have

$$S_n = (\alpha^n + \beta^n) + \left(\frac{\gamma^n - \delta^n}{\gamma - \delta}\right).$$

**Theorem 2.3.** Let $S_n$ denote the $n$-th numbers. Then the sum of first non-zero terms of $S_n$ is

$$\sum_{i=1}^{3} S_i = S_2 + 3S_{n-1} + 5S_{n-2} + 2S_{n-3} - 7.$$ 

(7)

**Proof.** From the recurrence relation of $S_n$, we get

$$2S_{n+1} + 2S_{n+2} = 3S_{n+3} + 2S_{n+4} + S_n.$$ 

(8)

Applying (7), we deduce that

$$2S_1 + 2S_2 = 3S_3 + 2S_4 + S_0,$$

$$2S_3 + 2S_4 = 3S_5 + 2S_6 + S_1,$$

$$2S_5 + 2S_6 = 3S_7 + 2S_8 + S_2,$$

$$\vdots$$

$$2S_{n-1} + 2S_{n-2} = 3S_{n+1} + 2S_{n+2} + S_n.$$ 

If we sum both sides of (8), then we have

$$2\sum_{i=1}^{3} S_i = S_2 + 3S_{n-1} + 5S_{n-2} + 2S_{n-3} - 2S_0 - 3S_1 - S_2 + S_1.$$ 

Thus we get

$$\sum_{i=1}^{3} S_i = \frac{S_2 + 3S_{n-1} + 5S_{n-2} + 2S_{n-3} - 7}{2}.$$ 

(9)

3. An Application of Quadra Lucas-Jacobsthal Numbers in Matrices

In this section, we will give some applications on matrix norms of Quadra Lucas-Jacobsthal numbers. Let $A = (a_{ij})$ be any $m \times n$ complex matrix. The spectral norm of the matrix $A$ are

$$\|A\|_2 = \sqrt{\max_{1 \leq \lambda \leq n} |\lambda(A^H A)|}$$

where $\lambda(A^H A)$ is an eigenvalue of $A^H A$ and $A^H$ is the conjugate transpose of the matrix $A$. 

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By a circulant matrix of order \( n \) is meant a square matrix of the form

\[
C = \text{Circ}(c_0, c_1, c_2, \ldots, c_{n-1}) = \begin{bmatrix}
c_0 & c_1 & c_2 & \cdots & c_{n-1} \\
c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_1 & c_2 & c_3 & \cdots & c_0
\end{bmatrix}.
\]

The eigenvalues of \( C \) are

\[
\lambda_j = \sum_{i=0}^{n-1} c_i w^{-ij},
\]

where \( w = e^{\frac{2\pi i}{n}} \) and \( i = \sqrt{-1} \).

**Lemma 3.1.** ([11]) Let \( A \) be an \( n \times n \) matrix with eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \). Then, \( A \) is a normal matrix if and only if the eigenvalues of \( A^T A \) are \( \| \lambda_1 \|, \| \lambda_2 \|, \ldots, \| \lambda_n \| \).

**Theorem 3.1.** The spectral norm of \( C = \text{Circ}(S_0, S_1, S_2, \ldots, S_{n-1}) \) is

\[
\| C \|_1 = \frac{S_{n-1} + 3S_{n-2} + 5S_{n-3} + 2S_{n-4} - 3}{2}.
\]

**Proof.** Since \( C \) is circulant matrix, from (10), for all \( t = 0, 1, 2, \ldots, n-1 \),

\[
\lambda_0(C) = \sum_{i=0}^{n-1} S_i (w^{-i})^t.
\]

Then for \( t = 0 \),

\[
\lambda_0(C) = \sum_{i=0}^{n-1} S_i.
\]

Hence, for \( 1 \leq m \leq n-1 \), we have

\[
\lambda_m = \left| \sum_{i=0}^{n-1} S_i (w^{-i}) \right| \leq \sum_{i=0}^{n-1} S_i | (w^{-i}) | \leq \sum_{i=0}^{n-1} S_i.
\]

By using the Lemma 3.1 and the fact that the matrix \( C \) is a normal matrix, we have

\[
\| C \|_1 = max_{0 \leq m \leq n-1} | \lambda_m | = max_{0 \leq m \leq n-1} | \lambda_0 |, \lambda_1 |, \ldots, | \lambda_{n-1} |.
\]

From (11), (12) and (13), we have

\[
\| C \|_1 = \frac{S_{n-1} + 3S_{n-2} + 5S_{n-3} + 2S_{n-4} - 3}{2}.
\]

**Definition 3.1.** The companion matrix of the Quadra Lucas-Jacobsthal recurrence relation is defined by

\[
A = \begin{bmatrix}
2 & -3 & -2 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

We note that \( \text{det}(A) = 2 \) and the eigenvalues of \( A \) are

\[
\lambda_1 = \alpha, \lambda_2 = \beta, \lambda_3 = \delta \text{ and } \lambda_4 = \gamma.
\]

**Definition 3.2.** ([11]) Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be the eigenvalues of a matrix \( A \). Then its spectral radius \( p(A) \) is defined as follows:

\[
p(A) = \max_{1 \leq i \leq n} | \lambda_i |.
\]

**Corollary 3.1.** Let \( A \) be the matrix of the Quadra Lucas-Jacobsthal recurrence relation. Then \( p(A) = 2 \).

4. References